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**Monetary Policy under Finite Speed of Trades and Myopia**

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# Monetary Policy under Finite Speed of Trades and Myopia

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**ABSTRACT.**— This paper provides a framework for monetary macro-policy, where the Central Bank intervenes on short-term loans markets, alternatively by manipulating interest rates or money supply. Trades are subject to a cash-in-advance constraint and to a size restriction. Agents, who are assumed to be myopic, can try to get around the restriction on size by retrading goods in a series of round. They can also borrow cash on the short term loan market. We show that in an appropriately defined strategic market game and under a gains-to-trade hypothesis: 1) trades will lead to a Pareto efficient allocation but only if the number of rounds of trade is infinite; 2) the monetary exchange process is pinned down uniquely in both its real and nominal variables; 3) the money supply should grow sufficiently fast to prevent the trading process to get stuck in a local liquidity trap; 4) if the Central Bank follows the Taylor rule, the trading process is bound to fall into a local liquidity trap.

**Keywords:** Strategic Market Game, Central Bank, Gains to trade, Taylor rule, Myopia, Liquidity trap, Finite Speed of trade.

**JEL Classification Numbers :** D50, E40, E50, E58.

## 1 Introduction

This paper addresses the issue of designing an optimal policy within a monetary dynamic general equilibrium set-up with myopic households, where the

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Bank can equivalently choose to monitor short-term rates or the corresponding money supply. Our main conclusion is that, whatever being the initial quantity of outside cash, an unbounded growth of money supply on the short-term loans market is unavoidable if the economy is to escape from any (local) liquidity trap. By itself, this result might not render the “orthodox” wisdom on inflation-targeting invalid. On the contrary, we even show that the monetary policy on short-term rates suffices to drive a globally unique trading process (both in nominal and real terms), confirming the standard opinion according to which inflation-targeting may indeed suffice to drive inflation. Our finding on the necessary unbounded growth of money supply is also in accordance with the huge, empirically observed growth rate of the world-wide  $M_0$  aggregate: 15% per year on average, since 2000, and 30% since 2008.

Our result on the necessary explosion of monetary supply, however, stands sharply in contrast with the now classical literature devoted to the Taylor rule once one realises that, in our model, an “optimal” monetary policy *must* induce an unbounded *inflation* as measured by the GDP deflator).<sup>1</sup> As a consequence, a literal application of Taylor rule would require the Central Bank to reduce the quantity of money injected into the economy, necessarily leading the latter into a (local) liquidity trap.

### 1.1. Myopic retrading with money

To make these points, we consider a cash-in-advance economy with heterogeneous households and several consumption goods. The cash-in-advance constraint à la Clower is only used for clarity. As emphasized in Remark 2.5 infra, we could as well rephrase our all set-up in terms of bid-ask spreads in a fashion similar to Duffie (1990). On the other hand, the cash-in-advance constraint can be interpreted as the consequence of money being the unique medium of exchange among all the marketed goods. That commodities cannot trade directly against each other can be explained as a consequence of underlying transaction costs (see, e.g., Dubey & Geanakoplos (2003b, section 17)).<sup>2</sup> The presence of several commodities and heterogeneous households allows one to reach clear conclusions about economic welfare. The Central Bank is supposed to conduct a traditional monetary policy by controlling spot interest rates. Equivalently, the Bank’s policy can be viewed as monitoring the quantity of money injected on the short-term loan markets (see Remark 2.7 below).

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<sup>1</sup>The “Great Moderation” of commodity prices is not at odds with this theoretical conclusion. It might well be the case, indeed, that, during the 1990s, inflationary pressures have migrated from the commodity markets to financial markets (where average returns are much higher). So that the expansionist monetary policy of Central Banks might have fed the recursive financial bubbles which have grown and burst since twenty years. In the stylized economy considered in this paper, we do not introduce financial assets. In a companion paper, we show the migration of inflation from commodity to financial markets.

<sup>2</sup>Following Dubey & Geanakoplos (2003b), we could allow *certain* commodities (but not all) to trade among each other. At the cost of reformulating the gains-to-trade assumption in terms of sufficiently missing links among commodities in comparison to the ratio between “outside” and “inside” money, our main results would remain intact.

We depart from the standard General Equilibrium (GET) framework by imposing an upper-bound on the size of permitted trades: Each time period, households can perform only small-scale trades. This is tantamount to assuming that the speed of trades is bounded. (By contrast, standard GET implicitly assumes the speed of trades to be infinite as any order of any size can be a priori performed.) As agents cannot reach their optimal consumption within one single round of trades (as they would in standard GET), this opens the room for retrading. Thus, in this paper, households can (re)trade the same storable goods during a finite number,  $T$ , of rounds. Clearly, if there are enough trading rounds, any order of any size can be replicated by means of several rounds of small-size orders. It therefore seems, at first glance, that our finite-speed restriction, besides being more realistic than the infinite-speed framework of standard GET, should not induce dramatic changes. It does, however, have a dramatic effect since it reduces the set of equilibrium paths to a singleton, not only in real, but also in nominal variables.<sup>3</sup>

Our second departure with intertemporal GET is that we assume agents to be myopic: Each of them trades, each period, so as to maximize her current utility in the short-run. The economic *rationale* for such a myopic behavior is not new: Even chess international grandmasters do not calculate more than four or five moves ahead, and it has been argued that, under quite reasonable circumstances and unless one is ready to assume that everybody shares perfect foresight, seeing further into the future does not mean seeing better.<sup>4</sup> On the other hand, there is a large body of evidence that agents forecasts are not consistent with the paradigm of rational expectations.<sup>5</sup> We show that, if there are enough gains-to-trade via money, trades will asymptotically vanish near some Pareto point. Therefore, the dynamics of this paper belongs to models of transition-to-equilibrium in line with Smale (1976), Champsaur & Cornet (1990), Bottazzi (1994), Giraud & Tsomocos (2010) and Bonnisseau & Orntangar (2010).<sup>6</sup>

There are two sources of money: In the initial period, some outside money is inherited from an unmodelled past;<sup>7</sup> then, at every subsequent time, short-term loan markets open allowing households to borrow some additional amount of money, at some short-term nominal rate  $r(t)$  clearing the money market. That is, there is no outside money in our model: All the available money is inside, and has been created by the Central Bank (whose budget is

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<sup>3</sup>Magill and Quinzii (1992) proposed another monetary theory, in which there is no outside money and yet equilibrium is locally —not globally— unique. They accomplished this, however, by assuming forced sales of goods: Each agent must sell all his goods to the government, which has an exogenously fixed stock  $M$  to pay in exchange. Here, nobody is forced to sell anything to anyone, and yet we get global uniqueness.

<sup>4</sup>See, e.g., Gray & Geanakoplos (1991) and the literature therein.

<sup>5</sup>See, e.g., Roberts (1997), Forsells and Kenny (2002), Adam and Padula (2009), to name but a few.

<sup>6</sup>See also Farmer & Geanakoplos (2008) for a plea in favor of transition-to-equilibrium models.

<sup>7</sup>It can be interpreted as public debt, fiscal transfers, long-term private debt, etc.

balanced at the end of period  $T$ ). This departs from the approach developed by Dubey & Geanakoplos (2003a) or Giraud & Tsomocos (2010), in which households were assumed to own some pocket money, free and clear of any debt.

An important and realistic feature of our model is that all exchange must be physically carried out between two instruments. If a household wants to buy a house with money, then it must turn over the money. This leads naturally to the idea that market actions form prices. The price of good  $\ell$  in terms of money is simply the total amount of money chasing  $\ell$  at the market, giving rise to a strategic market game.<sup>8</sup> Since we work with a continuum of agents, we follow Dubey & Geanakoplos (2003a) in recasting the monetary equilibria in terms of more familiar budget sets where individuals regard prices and interest rates as fixed. But the fundamental aspect of a game, that every choice of players' strategies engenders an outcome, is fully honored. When compared with Dubey & Geanakoplos (2003a), we even go one step further in this direction by imposing that, at a monetary equilibrium, whenever no commodity is supplied by the players' actions, then nobody trades. This is reminiscent of the well-known autarkic Nash equilibrium which survives in most of the strategic market games studied in the literature.<sup>9</sup> The advantage of doing so is that we recover existence of our extended monetary equilibrium even when standard monetary equilibrium à la Dubey-Geanakoplos fails to exist. The non-existence of standard monetary equilibria shows up as a local liquidity trap in terms of our extended solution concept.

### *1.2. Money non-neutrality and Taylor rule*

We show that the quantity of money injected in the short-run loan market by the Bank must increase at some rate which essentially depends upon the interplay between cash balances and gains-to-trade. To be more precise, at every  $\tau$ -local monetary equilibrium, spot interest rates must stand below the ratio between outside and inside cash balances. Moreover, we provide a discrete-time counterpart of the result obtained in Giraud & Tsomocos (2010): a short-term oriented monetary policy will induce a unique monetary trading process. As uniqueness obtains not only for real variables (as in Bonnisseau & Orntangar (2010)) but also for nominal ones (as in Giraud & Tsomocos (2010)), this partly confirms the traditional wisdom: At first glance, Inflation targeting seems to make sense since comparative dynamics can be performed (not only locally but even globally). Thus, our results rather confirm standard fiscal theory (cf. Woodford (2003)). Any change in outside money, however, induces a non-trivial change in the corresponding monetary path. This parameterization of trading paths with outside money reconciles our result with the strong indeterminacy obtained in other models with no outside money (cf. Drèze & Polemarchakis (2000), Bloise *et alli* (2005),

<sup>8</sup>See Giraud (2003) for an introduction, as well as the whole special issue.

<sup>9</sup>See Weyers (2003) for an attempt to get rid of this autarkic outcome by refining the strategic equilibrium concept.

Bloise & Reichlin (2008). This indeterminacy can therefore be interpreted as equivalent to the existence of an infinite number of outside cash distributions (e.g., long-term debt levels, public deficits, fiscal transfers, etc.) that are compatible with our equilibrium conditions.

The lesson to draw from our paper, however, is not “monetarist” in essence: we show that money is non-neutral, neither in the short-, nor in the long-run. We stress that, at variance with standard results obtained in the so-called “New-Keynesian” model,<sup>10</sup> money has real effects even though there are no nominal rigidities. On the other hand, for a monetary equilibrium to involve effective trades, the quantity of circulating money must evolve according to a measure of gains to trade (first introduced by Dubey & Geanakoplos (2003a)). That is: Monetary expansion must be related to the “real” sector of the economy – which contrasts with both standard inflation-targeting and money-targeting policies. Finally, we also prove that, for a monetary trading process to converge to some Pareto-efficient rest-point, liquidity must increase in such a way that spot interest rates remain *lower* than current gains-to-trade. A consequence is that, as the economy is approaching the Pareto curve, short-term rates must converge to zero. Otherwise stated: Liquidity must blow up at infinity. And so do commodity prices along the unique monetary trade path.

The consequences for the traditional approach of monetary policy in terms of Taylor’s rule are striking, though intuitively quite simple. Most macro-economic models in which monetary policies are discussed rely on the fiction of a single representative household or a single consumption commodity. In both cases, efficiency issues disappear and the measure of gains-to-trade is trivially constant. Therefore, a condition in terms of gains-to-trade such as the one we emphasize here (borrowed from Dubey & Geanakoplos (2003a)) could not emerge. On the other hand, a naive implementation of Taylor’s rule in our multiple-commodity world populated by heterogeneous households would inevitably lead the economy to a (local) liquidity trap. As soon as prices increase, indeed (and we show that if the economy is to escape from any trap, prices must increase), the Bank would raise its short-term rate (instead of reducing it, as this paper recommends). Sooner or later, this will leave short-term nominal interest rates above the threshold of current gains-to-trade, and brings the economy into a liquidity trap, from which the sole road out rests on a significant increase in the amount of circulating money. This point might provide a plausible explanation of the Fed’s recent quantitative easing policy. It stands in sharp contrast with the conclusion obtained, for instance, by Molnar & Santoro (2005) on optimal monetary policy when households don’t conform to the rational expectations paradigm but are still learning. There, indeed, it turns out that an optimal monetary policy should be more aggressive when agents don’t follow rational expectations than when they do. Here, we show that, when agents are entirely myopic, whether

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<sup>10</sup>See Clarida, Gali, and Gertler (1999) for an early survey, and Woodford (2003) for a synthesis.

aggressive or not, a standard rule such as the one recommended by Taylor is inefficient.

The paper is organized as follows: the next section develops our discrete-time monetary exchange process. Section 4 sets forth the main result of the paper and provide some preliminary results as a preparation for the proof that is given in the last section. More technical material on linear monetary economies is relegated to the Appendix. This material is interesting in its own right and plays a pivotal role in the proof of our main result (see especially Lemma 2.2.).

## 2 Monetary economy

Let us start with a static, Arrow-Debreu exchange economy with  $L \geq 1$  commodities and  $N \geq 1$  heterogenous households (each of them being represented by a continuum of unit length of identical clones).<sup>11</sup> The consumption sets are  $\mathbb{R}_+^L$ , preferences of consumer  $i$  are represented by a utility function  $u_i: \mathbb{R}_+^L \rightarrow \mathbb{R}$ , and the initial vector of endowments is denoted  $\mathbf{e}_i \in \mathbb{R}_+^L$ . An individual  $i$  is endowed with cash balances  $\mu_i \geq 0$  (acquired by different ways to be detailed later on in the paper). If he wishes so,  $i$  can also borrow some additional cash,  $m_i$ , at the short-term nominal rate  $r$ . Let  $M$  denote the total amount of money injected by the Bank on the loan market, and  $\bar{\mu} := \sum_i \mu_i$ , the aggregate “outside money”. Individual  $i$  then can purchase commodities under the cash-in-advance constraint:

$$p \cdot z_i^+ \leq \mu_i + m_i, \quad (1)$$

where  $z_i := x_i - \mathbf{e}_i$  stands for the net trade of  $i$  (where  $\mathbf{e}_i$  is an exogenously given initial endowment of consumption goods). He also accumulates end-of-period balances through receipts from the sale of commodities,  $p \cdot z_i^-$ . At the end of a trade round,  $i$  has to be able to pay back his debt,  $(1+r)m_i$ . Hence, he settles his current debt according to:

$$\Delta(1) + p \cdot z_i^- \geq (1+r)m_i. \quad (2)$$

where  $\Delta(1)$  stands for the difference between the right- and the left-hand sides of (1). Taken together, (1) and (2) imply :

$$p \cdot z_i \leq \mu_i - rm_i. \quad (3)$$

A *monetary economy* is then characterized by its monetary parameters  $M$ ,  $(\mu_i)_i$ , and will be denoted by:

$$(\mathcal{E}(\mathbf{e}), (\mu_i), M)$$

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<sup>11</sup>In  $\mathbb{R}^L$ ,  $S$  denotes the unit simplex and  $\|x\| = \sum_{h=1}^L |x_h|$  for all  $x \in \mathbb{R}^L$ . For any given vector  $x$ ,  $x^+$  (resp.  $x^-$ ) stands for the vector whose components are  $x_k^+ := \max\{0, x_k\}$  (resp.  $x_k^- := \max\{0, -x_k\}$ ).



where  $\mathcal{E}(\mathbf{e}) := (\mathbb{R}_+^\ell, u_i, \mathbf{e}_i)_{i=1}^m$  is the underlying “real” exchange economy. We now define a monetary equilibrium. At such an equilibrium, agents regard macrovariables (prices,  $p$ , and interest rate,  $r$ ) as fixed (i.e., uninfluenced by their own actions). The action of player  $i$  consists in borrowing money,  $m_i$ , offering a vector,  $z_i^- \in \mathbb{R}_+^L$ , for sale, in exchange for  $z_i^+ \in \mathbb{R}_+^L$ . Given an action profile, the outcome,  $(x_i)_i \in (\mathbb{R}_+^L)^N$ , is the final allocation of households.

**Definition 2.1**  $(p^*, r^*, (m_i^*, z_i^*, x_i^*)_i) \in \mathbb{R}_{++}^L \times \mathbb{R}_+ \times (\mathbb{R}_+ \times \mathbb{R}^L \times \mathbb{R}_+^L)^m$  is an *extended monetary equilibrium* (EME for short) of the monetary economy  $(\mathcal{E}(\mathbf{e}), (\mu_i), M)$  if

- (i) For all  $i = 1, \dots, N$ ,  $(z_i^*, m_i^*)$  is a solution of 
$$\begin{cases} \text{maximize } u_i(z_i + \mathbf{e}_i) \text{ s.t.:} \\ p^* \cdot z_i^+ \leq \mu_i + m_i \\ p^* \cdot z_i \leq \mu_i - r m_i \\ m_i \geq 0, z_i \geq -\mathbf{e}_i. \end{cases}$$
- (ii) If  $\sum_i z_i^{*-} = 0$ , then  $m_i^* = 0$  and  $x_i^* = \mathbf{e}_i$  for every  $i$ , while  $(r, p) \in \mathbb{R}_+^{L+1}$  are arbitrary.
- (iii) Else,  $x_i^* = \mathbf{e}_i + z_i^* \forall i$ , and  $\sum_{i=1}^N m_i^* \leq M$ ;  $\sum_{i=1}^N (x_i^* - \mathbf{e}_i) \leq 0$ .

Note that, whenever  $\mu_i = m_i^* = 0$ , then,  $i$  cannot trade:  $z_i^{*+} = z_i^{*-} = 0$ . A EME with no trade at all (i.e.,  $x_i^* = \mathbf{e}_i$ , all  $i$ ) is called *autarkic*. Otherwise, it is said to be *effective*. At an autarkic EME, money has no value, and its market “price” is arbitrary.

**Remark 2.1** If one replaces condition (ii) by  $x_i^* = \mathbf{e}_i + z_i$  for *every* action profile, then we get the definition of a monetary equilibrium (ME) as introduced by Dubey & Geanakoplos (2003a). In particular, every effective EME is a ME in the sense of Dubey & Geanakoplos (2003a), and vice-versa. Instead, our condition (ii) says that, whenever the players’ actions lead to a null aggregate supply, nobody trades. This condition seems natural, especially if we think in terms of some underlying strategic market game (with each type  $i$  being represented by a continuum of identical agents).<sup>12</sup> Indeed, in such a game, the action of player  $i$  would consist, say, in some bid  $b_i^k$  and supply,  $q_i^k$ , for each commodity  $k = 1, \dots, L$ . Prices would given by

$$p_k = \frac{\sum_i b_i^k}{\sum_i q_i^k} \mathbf{1}_{\{\sum_i q_i^k > 0\}},$$

where  $\mathbf{1}_{\{\sum_i q_i^k > 0\}} = 1$  if the condition,  $\sum_i q_i^k > 0$ , is fulfilled,  $= 0$  otherwise. The final allocation of player  $i$  would be

$$x_i^k = \mathbf{e}_i^k - q_i^k + \frac{b_i^k}{p_k} \mathbf{1}_{\{p_k > 0\}}.$$

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<sup>12</sup>See Giraud (2003) for an introduction.

In such a game, no-trade is always a Nash equilibrium, how far initial endowments are from being Pareto-optimal : if every player supplies no commodity and bids nothing ( $q_i^k = b_i^k = 0$ , every  $i$  and every  $k$ ), then there is nothing to be traded, and it is a best-reply to refrain from supplying anything. Here, following Dubey & Geanakoplos (2003a), we define the action,  $(z_i^+, z_i^-)$ , of a player in a somewhat different way. The relationship with the rules just described is obvious ( $z_i^+ = q_i$  and  $p_k z_i^-(k) = b_i^k$ ), but, by doing so, we can get rid of the autarkic Nash equilibrium whenever gains-to-trade are sufficient to provide the needed incentive for players to trade together. On the other hand, as we shall see, by adding (ii), we recover no-trade as an eME whenever gains-to-trade are too low (with respect to the ratio between outside and inside money). The absence of condition (ii) in the definition of monetary equilibria in Dubey & Geanakoplos (2003a) is responsible for their existence failure when gains-to-trade are low. For the ease of notations, we simply call eME our extended monetary solution concept.

**Remark 2.2** So far, we assumed that the Central Bank is committed to inject some quantity,  $M$ , of inside money, while interest rate,  $r$ , forms endogenously so as to clear the loan market. As in Dubey & Geanakoplos (2003a,b), we could conversely assume that the Central Bank fixes the rate  $r$  and stands ready to buy bank bonds in order to clear the loan market. More precisely, every effective eME with  $M > 0$  can be translated into an equilibrium where  $r = \bar{\mu}/M$  is exogenous and  $M$ , endogenous. Reciprocally, every equilibrium with some exogenous  $r > 0$  is equivalent to an eME with  $M = \bar{\mu}/r$  exogenous.

There is no loss of generality in assuming that no agent has the null endowment and every marketed commodity is actually present in the economy (i.e.,  $\mathbf{e}_i > 0$ , every  $i$ , and  $\sum_i \mathbf{e}_i \gg 0$ ). We posit the following weak assumptions on the “real” sector.

**Assumption (C).** For all  $i$ ,  $u_i(\cdot)$  is continuous, concave and strictly increasing on  $\mathbb{R}_+^L$ .

**Remark 2.3** Under Assumption (C), at an effective eME, constraints (1) and (3) are binding and the following holds:

- (i) For each  $i$ ,  $m_i^* = \frac{p^* \cdot z_i^{*-}}{1 + r^*}$ , i.e., each household borrows exactly the amount of inside money it can afford thanks to its sale recipes.
- (iii) When binding, the constraints (1) and (3) are equivalent to:

$$p^* \cdot z_i^{*+} = \mu_i + m_i^* \quad \text{and} \quad p^* \cdot z_i^{*-} = (1 + r^*)m_i^* \quad (4)$$

We shall use the following notation for every commodity price vector  $p$  and interest rate  $r$  :  $(p, r) \cdot (x_i, m_i) := p \cdot x_i + r m_i$ .

**Remark 2.4** Suppose that no initial endowment of money is available in the economy, i.e.,  $\bar{\mu} := \sum_i \mu_i = 0$ . Then, every EME reduces to a standard Walras equilibrium as soon as  $r = 0$ . Indeed,  $r = 0$  implies, for each  $i$ :

$$p \cdot z_i^+ = \frac{p \cdot z_i^-}{1+r} = p \cdot z_i^-$$

so that the monetary budget constraints reduce to the standard ones. Of course, prices are then defined only up to a positive constant as in the usual, barter case.<sup>13</sup> On the other hand, if  $r > 0$ , autarky is the unique (extended) monetary equilibrium allocation. This time, indeed,

$$p \cdot z_i^+ = \frac{p \cdot z_i^-}{1+r} < p \cdot z_i^-$$

so that, if  $\sum_i z_i^- > 0$ , then  $p \cdot \sum_i z_i < 0$  — which contradicts the feasibility,  $\sum_i z_i = 0$ .

Let us end this subsection with an alternate characterization of each agent's budget set. The next Lemma will be handfull in the sequel, as it allows to get rid of the non-differentiability of the cash-in-advance constraint.

**Lemma 2.1** *For every  $(p, r) \in \mathbb{R}_+^{L+1}$  and  $(m_i, \mu_i) \in \mathbb{R}_+^2$ , the three following sets of constraints are equivalent:<sup>14</sup>*

- (a)  $p \cdot z_i^+ \leq m_i + \mu_i$  and  $p \cdot z_i \leq \mu_i - rm_i$ ;
- (b)  $p \cdot z_i^+ - \frac{1}{1+r} p \cdot z_i^- \leq \mu_i$ ;
- (c) (0)  $p \cdot z_i \leq \mu_i$  and (k)  $p_k \cdot z_i^k + \frac{1}{1+r} p^{-k} \cdot z_i^{-k} \leq \mu_i$  for every  $k = 1, \dots, L$ .

**Proof of Lemma 2.1.**

- (a)  $\iff$  (b) is Lemma 1 in Dubey & Geanakoplos (2003a).
- (b)  $\Rightarrow$  (c). Take  $x_i$  such that  $z_i = x_i - \mathbf{e}_i$  verifies (b). First,

$$p \cdot z_i \leq p \cdot z_i^+ - \frac{1}{1+r} \cdot z_i^- \leq \mu_i.$$

Hence, constraint (0) is verified. Second,

$$\begin{aligned} p_k z_i^k + \frac{1}{1+r} p^{-k} \cdot z_i^{-k} &\leq p_k z_i^k + p^{-k} \cdot [z_i^{-k}]^+ - \frac{1}{1+r} p^{-k} \cdot [z_i^{-k}]^- \\ &\leq p \cdot z_i^+ - \frac{1}{1+r} p \cdot z_i^- \\ &\leq \mu_i. \end{aligned}$$

So that every constraint (k) is fulfilled,  $k = 1, \dots, L$ .

<sup>13</sup>The reason why we do not recover the real indeterminacy of Drèze & Polemarchakis (2005) is that the profits of the central Bank are not redistributed to the citizens. This point is made precise in section 3 below.

<sup>14</sup>For  $p = (p_1, \dots, p_L)$ ,  $p^{-k}$  denotes the  $(L-1)$ -vector  $(p_1, \dots, p_{k-1}, p_{k+1}, \dots, p_L)$ .

(c)  $\Rightarrow$  (b) The budget set,  $B(c)$ , induced by constraint (c) is a convex and compact polyhedron. Its extreme points are given by the following allocations:

- Suppose that  $i$  does not sell any item (i.e.,  $z_i^- = 0$ ) and sells all her cash,  $\mu_i$ , for the purchase of commodity  $k \in \{1, \dots, L\}$ . This provides  $L$  extreme points which all verify (b) since:

$$p \cdot z_i^+ - \frac{1}{1+r} p \cdot z_i^- = p \cdot z_i \leq \mu_i.$$

- Next, suppose that  $i$  sells a positive amount of her initial endowment. The remaining extreme points of the budget set induced by (b) correspond to the  $L$  allocations where  $i$  buys *only* good  $k$  (and sells all her other goods in order to finance that purchase), i.e.,  $z_i^k > 0$  and  $z_i^{-k} = -\mathbf{e}_i^{-k}$ . Then,

$$\begin{aligned} p \cdot z_i^+ - \frac{1}{1+r} p \cdot z_i^- &= p_j z_i^k - \frac{1}{1+r} p_j [z_i^{-k}]^- \\ &= p_j z_i^k + \frac{1}{1+r} p_j z_i^{-k} \\ &\leq \mu_i, \end{aligned}$$

so that each of these additional  $L$  extreme points verifies (b). Thus, the budget set,  $B(b)$ , induced by (b) (also a convex polyhedron) contains all the extreme points of  $B(c)$ . It therefore contains  $B(c)$ .  $\square$

A corollary of formulation (b) in Lemma 2.1 is that, if  $r = \mu_i = 0$ , each  $i$ , then a EME coincides with a Walrasian equilibrium. This is so, in particular, if  $M > 0$  and  $\bar{\mu} = 0$ .

**Remark 2.5** Another consequence is that our approach in terms of a cash-in-advance constraint à la Clower could be equivalently written in terms of a bid-ask spread between buying and selling prices: This wedge is given by  $p - \frac{1}{1+r}p$ , i.e., by the ratio,  $\frac{r}{1+r}$ , between buying and selling prices. This remark reconciles the cash-in-advance viewpoint with the one defended by Foley (1970), Hahn (1971) or Duffie (1990).

## 2.1 Gains-to-trade

Let us now recall the measure,  $\gamma(\cdot)$ , of local gains-to-trade (see Dubey & Geanakoplos (2003a)). Let  $z_i \in \mathbb{R}^L$  be a net trade vector of  $i$ , with positive component representing purchases and negative ones representing sales. For any scalar  $\gamma \geq 0$ , define the vector  $z_i(\gamma) \in \mathbb{R}^L$  whose  $k^{\text{th}}$  component is given by:

$$z_i^k(\gamma) := \min \left\{ z_i^k, \frac{z_i^k}{1+\gamma} \right\} \quad k = 1, \dots, \ell. \quad (5)$$

The vector  $z_i(\gamma)$  entails a diminution of purchases in  $z_i$  by the fraction  $1/(1+\gamma)$ . There are *gains to  $\gamma$ -diminished trades* in  $\mathcal{E}$  if there exist net trades  $(z_i)_i$  that are feasible (i.e., such that  $\sum_i z_i = 0$ ), and such that  $u_i(\mathbf{e}_i + z_i(\gamma)) \geq u_i(\mathbf{e}_i)$  for all  $i$  with at least one inequality being strict. In words, it should be possible for households to Pareto-improve on no-trade in spite of the  $\gamma$ -handicap on trades. The measure  $\gamma(\mathbf{e})$  is the supremum of all handicaps that permit Pareto-improvement. Clearly,  $\mathbf{e}$  is Pareto-optimal if, and only if,  $\gamma(\mathbf{e}) = 0$ . Notice that gains-to-trade depend only on the “real” sector of  $\mathcal{E}$ , and not on its monetary sector.

As in Dubey & Geanakoplos (2003a) or Giraud & Tsomocos (2010), the following is a key assumption for guaranteeing that money has a positive value at a eME (hence, for solving Hahn’s celebrated paradox).

### Gains-to-trade hypothesis (GT).

The aggregate quantity of inside money,  $M$ , verifies:

$$\frac{\bar{\mu}}{\gamma(\mathbf{e})} < M. \quad (6)$$

**Theorem 2.1** *Under (C),*

(i) *Every effective eME verifies:*

$$r = \bar{\mu}/M, \quad (7)$$

*and the following version of the Quantity Theory of Money holds:*

$$M + \bar{\mu} = p \cdot \sum_i z_i^+. \quad (8)$$

(ii) *If (GT) is in force, the monetary economy  $(\mathcal{E}(\mathbf{e}), (\mu_i)_i, M)$  admits an effective eME,  $(p, r, (z_i, x_i, m_i))$ .*

(iii) *When  $M \rightarrow +\infty$ , then, at the limit, an effective eME allocation coincides with a Walrasian equilibrium allocation of the “real” economy,  $\mathcal{E}(\mathbf{e})$ .*

### Proof of Theorem 2.1

(i) At an effective eME, each budget constraint,  $p \cdot z_i \leq \mu_i - r m_i$ , is binding (see Remark 2.3). Summing over  $i$  yields:  $0 = p \cdot \sum_i z_i = \bar{\mu} - r \sum_i m_i = \bar{\mu} - r M$ . Hence,  $r = \bar{\mu}/M$ . Similarly, each individual cash-in-advance constraint must be binding. Summing over  $i$  yields (8).

(ii) Suppose (GT) is in force. We follow the proof of Theorem 2 in Dubey & Geanakoplos (2003a) with a small change at the end. Introduce an external player who places  $\varepsilon > 0$  on every market. As a consequence, our definition of an extended monetary equilibrium reduces to the one adopted in Dubey & Geanakoplos (2003a). Prove the existence of a  $\varepsilon$ -eME by a standard fixed-point argument on the space of action profiles. When letting

$\varepsilon \rightarrow 0^+$ , two cases occur. Either prices,  $p(\varepsilon)$ , remain bounded — in which case, by extracting a subsequence, one shows that the limit is a eME. Or, prices are unbounded. In this latter case, whenever trades are effective for every  $\varepsilon > 0$ , then,  $\gamma(x(\varepsilon)) \leq r_\varepsilon = \bar{\mu}_\varepsilon/M_\varepsilon$  according to Theorem 4 in Dubey & Geanakoplos (2003a). It follows from Lemma 3 in this paper that  $\gamma(\cdot)$  is a lower-semi-continuous function, so that the last inequality passes to the limit:  $\gamma(\mathbf{e}) = \gamma(x(0)) \leq \liminf_{\varepsilon \rightarrow 0^+} \gamma(x(\varepsilon)) \leq \lim_{\varepsilon \rightarrow 0^+} r(\varepsilon) = r(0) = \bar{\mu}/M$  — which contradicts (GT).

(iii) The proof is essentially the same as in Dubey & Geanakoplos (2003a), and is given only for the sake of completeness. Consider a sequence of effective eME prices,  $(p(k))_k$  associated to the sequence  $(m_i(k))_{i,k} \rightarrow_{k \rightarrow \infty} \infty$ . The corresponding sequence,  $x(k) = \mathbf{e} + z(k)$ , of allocations lies within the (compact) feasible set  $(\sum_i z_i(k) = 0)$ . Normalize each price vector

$$q(k) := \frac{1}{\sum_\ell p_\ell(k)} p(k).$$

The limit price (of some subsequence)  $q = \lim_{k \rightarrow \infty} q(k)$  supports the limit eME-allocation,  $x := \lim_{k \rightarrow \infty} x(k)$  as a Walrasian equilibrium. Indeed, suppose, first, that  $\bar{\mu} = 0$ . Then,  $r = 0$  and formulation (b) of each individual budget constraints in Lemma 2.1 shows that each household's budget set reduces to the Walrasian one. Now, if  $\bar{\mu} > 0$ , the same argument holds once it is realized that net trades,  $(z_i)$ , are uniformly bounded by initial endowments. Hence, (8) implies that prices must explode to infinity,  $p(k) \rightarrow \infty$  while  $r(k) \rightarrow 0^+$ . The purchasing power of cash,  $\mu_i$ , hence tends to zero, and we are back to the previous situation.  $\square$

The intuition is as follows: If the gains-to-trade assumption is in force, money has value and effective trades will take place. That is, money must be available at a sufficiently low cost,  $r$ , for it to correctly play its role as a “grease that turns the wheels of commerce”. The threshold above which  $r$  induces too much friction for money to have a positive value (and for the economy to escape from the autarkic outcome) is not independent from the real sector of the economy: It turns out to be given precisely by the measure of gains-to-trade. On the other hand, at an effective equilibrium, the interest rate is given by the ratio,  $\bar{\mu}/M$ , between (aggregate) “outside” and “inside” money. This provides a lower-bound on the quantity of money,  $M$ , that must be injected by the Bank for markets to function correctly. Indeed, (7) means that, at equilibrium, (GT) is equivalent to  $\bar{\mu}/M > r^*$ .

**Remark 2.6** Notice also that, as long as we are only concerned with existence, strict monotonicity of preferences is not needed. A weaker notion (the having-wanting assumption) would suffice, as shown by Dubey & Geanakoplos (2003a). At first glance, this comes somehow as a surprise, since strict monotonicity serves as well, among other things, as a substitute for the standard, but utterly unrealistic, survival assumption ( $\mathbf{e}_i \gg 0$ ). Indeed, it is well known (cf. e.g., McKenzie (1959, 1961)) that strict monotonicity can replace

the survival assumption provided that each household owns at least one commodity (i.e.,  $\mathbf{e}_i > 0 \forall i$ ), as these two conditions ensure that  $p \cdot \mathbf{e}_i > 0 \forall i$ . In our monetary set-up, however, a non-trivial amount of initial owned cash,  $\mu_i > 0$ , suffices to ensure that agent  $i$  has some non-trivial wealth when entering the market. So that the survival assumption is not necessary anyway.

## 2.2 Monetary local economy

In the next section, we shall suppose that households can trade (and re-trade) the same  $L$  long-lived commodities.<sup>15</sup> Moreover, at each time period, the volume of trades will be bounded by some (exogenously given) physical/institutional limit, captured by means of a parameter,  $\tau$ . The next building block for our dynamics is the study of  $\tau$ -local monetary equilibria, to which we now turn.

**Definition 2.2** For all  $\tau \in (0; 1]^N$ ,  $(p^*, r^*, (m_i^*, z_i^*, x_i^*))$  is a  $\tau$ -local extended monetary equilibrium of  $(\mathcal{E}(\mathbf{e}), (\mu_i), M)$  if

$$(i) \text{ For all } i, (z_i^*, m_i^*) \text{ is a solution of } \begin{cases} \text{maximize } u_i(z_i + \mathbf{e}_i) \text{ s.t.} \\ p^* \cdot z_i^+ = \tau(\mu_i + m_i) \\ p^* \cdot z_i^- = \tau(1 + r)m_i \\ z_i \geq -\tau \mathbf{e}_i \text{ and } m_i \geq 0. \end{cases}$$

(ii) If  $\sum_i z_i^{*-} = 0$ , then  $m_i^* = 0$  and  $x_i^* = \mathbf{e}_i$  for every  $i$ .

(iii) Else,  $x_i^* = \mathbf{e}_i + z_i \forall i$ , and  $\sum_{i=1}^N \tau m_i^* \leq M$ ;  $\sum_{i=1}^N (x_i^* - \mathbf{e}_i) \leq 0$ .

A  $\tau$ -local eME is a monetary equilibrium of a monetary  $\tau$ -local economy  $(\mathcal{E}^\tau(\mathbf{e}), (\mu_i), M)$  in which each agent spends at most a fraction  $\tau$  of her money for purchase purposes, and where net trades are forced to stay within a neighborhood of zero (whose size depends upon  $\tau$ ). Note that, when  $\tau = 1$  (resp.  $\tau = 0$ ), for every  $i$ , we are back to a standard eME (resp. we have an autarkic equilibrium). The parameter  $\tau$  may depend upon  $i$ , reflecting the differences between agents in their participation to markets.

As  $\tau$  is close to 0, the size of permitted trades in a  $\tau$ -local eME shrinks to 0. Thus, the shape of (concave) preferences becomes close to their first-order linear approximation around  $\mathbf{e}_i$ . The next definition captures this intuition. As in Bonnisseau & Orntangar (2010), we associate to each monetary economy  $(\mathcal{E}(\mathbf{e}), (\mu_i), M)$  an auxiliary economy, denoted by  $(\hat{\mathcal{E}}^\tau(\mathbf{e}), (\mu_i), M)$ , obtained after having replaced each utility function,  $u_i(\cdot)$ , by an approximation of its variation rate:

$$u^\tau(x_i) = \frac{u_i(\mathbf{e}_i + \tau(x_i - \mathbf{e}_i)) - u_i(\mathbf{e}_i)}{\tau}. \quad (9)$$

<sup>15</sup>Thus, unlike the standard “Walrasian tâtonnement”, trades will take place before a rest-point has been reached.

**Remark 2.7** One easily checks that:

- 1) If  $(p^*, r^*, (x_i^*, m_i^*))$  is a  $\tau$ -local eME of  $(\mathcal{E}(\mathbf{e}), (\mu_i), M)$ , then  $(p^*, r^*, (\mathbf{e}_i + \tau_i^{-1}(x_i^* - \mathbf{e}_i), m_i^*))$  is a eME of  $(\mathcal{E}^\tau(\mathbf{e}), (\mu_i), M)$ .
- 2) Conversely if  $(p^*, r^*, (\xi_i^*, m_i^*))$  is a eME of  $(\mathcal{E}^\tau(\mathbf{e}), (\mu_i), M)$ , then  $(p^*, r^*, (\mathbf{e}_i + \tau(\xi_i^* - \mathbf{e}_i), m_i^*))$  is a  $\tau$ -local eME of  $(\mathcal{E}(\mathbf{e}), (\mu_i), M)$ .

For the next result, we need to strengthen Assumption (C) in the following classical way (see, e.g., Balasko (1988)):

**Assumption (D).**

- a) (*Smooth monotonicity*)  $u_i$  is  $\mathcal{C}^3$  on an open subset of  $\mathbb{R}^L$  containing  $\mathbb{R}_{++}^L$ , and  $\nabla u_i(x_i) \in \mathbb{R}_{++}^L$  for all  $x_i \in \mathbb{R}_{++}^L$ ;
- b) (*Differentiable strict quasi-concavity*) for all  $x_i \in \mathbb{R}_{++}^L$  and for all  $z_i \in \mathbb{R}^L \setminus \{0\}$ , one has  $[\nabla u_i(x_i) \cdot z_i = 0] \Rightarrow [z_i \cdot D^2 u_i(x_i)(z_i) < 0]$ ;

**Theorem 2.2** *Under Assumptions (D) and (GT), there exists  $\tilde{\tau} > 0$  sufficiently small so that, for all  $\tau \in (0, \tilde{\tau}]^N$ , the economy  $(\mathcal{E}(\mathbf{e}), (\mu_i), M)$  has exactly one effective  $\tau$ -local eME.*

We emphasize that, here, global uniqueness holds not only in “real” terms (trades and relative price ratios) but also in “nominal” ones (the absolute level of prices). This stands in contrast to Bonnisseau & Orntangar (2010) where global uniqueness obtains only in relative prices, and to Dubey & Geanakoplos (2003a) where, without any bound on monetary trades, nominal uniqueness obtains only locally and generically.

Let us prepare for the proof of Theorem 2.2 with the following Lemmata.

**Lemma 2.2** *There exists  $\bar{\tau} > 0$  so that, for every  $0 \leq \tau \leq \bar{\tau}$  and every  $\tau$ -local eME,  $(p, r, x, m)$ , if  $x_i^k > \mathbf{e}_i^k$  for some individual  $i$  and some good  $k$ , then:*

(a) *Either, at  $(p, r, x, m)$ ,  $i$  does not sell any item (i.e.,  $z_i \geq 0$ ), in which case:*

$$\frac{1}{p_k} \frac{\partial}{\partial x^k} u_i^\tau(\mathbf{e}_i + z_i) \geq \frac{1}{p_\ell} \frac{\partial}{\partial x^\ell} u_i^\tau(\mathbf{e}_i + z_i) \quad \forall \ell = 1, \dots, L. \quad (10)$$

(b) *else,  $z_i^\ell < 0$  for some good  $\ell \neq k$ , and*

$$\frac{1}{(1+r)p_k} \frac{\partial}{\partial x^k} u_i(\mathbf{e}_i + \tau(z_i)) \geq \frac{1}{p_\ell} \frac{\partial}{\partial x^\ell} u_i(\mathbf{e}_i + \tau(z_i)) \quad (11)$$

(ii) *In both cases, whenever the inequality is strict for some good  $\ell \neq k$ , then  $x_i^\ell = 0$ .*



**Proof of Lemma 2.2.**

According to Lemma 2.1, the individual maximization program is equivalent to:

$$(\mathcal{P}) = \begin{cases} \text{maximize}_{z_i} u_i^\tau(z_i + \mathbf{e}_i) \text{ subject to:} \\ p \cdot z_i \leq \mu_i \\ p_k \cdot z_i^k + \frac{1}{1+r} p^{-k} \cdot z_i^{-k} \leq \mu_i \quad k = 1, \dots, L \end{cases}$$

Karush-Kuhn-Tucker theorem implies that any solution  $x_i^*$  of  $(\mathcal{P})$  must therefore also solve the following linearized problem:

$$(\mathcal{Q}) = \begin{cases} \text{maximize}_\xi \nabla u_i^\tau(x_i) \cdot \xi_i \text{ subject to:} \\ p \cdot (\xi_i - x_i^*) \leq 0 \\ p_k \cdot (\xi_i^k - x_i^{*k}) + \frac{1}{1+r} p^{-k} \cdot (\xi_i^{-k} - x_i^{*-k}) \leq 0 \quad k = 1, \dots, L \end{cases}$$

Equations (25) and (26) in the Appendix (section 6.2) applied on  $(\mathcal{Q})$  imply that, if  $z_i^* > 0$  (i.e.,  $i$  does only uses her cash,  $\tau\mu_i$  to buy goods, and does not sell any item), then

$$\frac{\frac{\partial}{\partial x^k} u_i^\tau(x_i)}{p_k} \geq \frac{\frac{\partial}{\partial x^\ell} u_i^\tau(x_i)}{p_\ell} \quad \forall \ell = 1, \dots, L.$$

This, in turn, is equivalent to (10). Alternatively, if there exists some good  $\ell \neq k$  such that  $z_i^\ell < 0$ :

$$\frac{\frac{\partial}{\partial x^k} u_i^\tau(x_i)}{(1+r)p_k} \geq \frac{\frac{\partial}{\partial x^\ell} u_i^\tau(x_i)}{p_\ell},$$

and inequality (11) follows.

(ii) follows from (25) and (26) in the Appendix, applied on  $(\mathcal{Q})$ . □

The next lemma provides a lower-bound on equilibrium prices, independent from  $\tau$ .

**Lemma 2.3** *Let  $p^\tau$  be an eME price vector of  $(\mathcal{E}^\tau(\mathbf{e}), (\mu_i), M)$ . There exists  $\eta > 0$  (independent of  $\tau$ ) such that  $p_\ell^\tau > \eta$  for every  $\tau \in [0, 1]$ .*

**Proof of Lemma 2.3**

Let  $K > \sum_i \sum_\ell \mathbf{e}_\ell^i$ . Define  $u_i^* := \max_{\tau \in [0,1]} u_i^\tau(K, \dots, K)$ , fix some  $\tau \in [0, 1]$ , and let  $H^\tau > 0$  be chosen large enough so that

$$u_i^\tau(0, \dots, 0, H^\tau, 0, \dots, 0) > u_i^*,$$

for  $H^\tau$  in any component. The following argument (adapted from footnote 19 in Dubey & Geanakoplos (2003)) proves that such an  $H^\tau$  exists. Let  $\square$  be the cube with sides of length  $K$  in  $\mathbb{R}_+^L$ . Define  $\tilde{u}_i^\tau : \mathbb{R}_+^L \rightarrow \mathbb{R}$  by  $\tilde{u}_i^\tau(y) := \inf\{L_x(y), x \in u, L_x \text{ is an affine function representing the supporting hyperplane to the graph of } u_i^\tau \text{ at the point } (x, u_i^\tau(x))\}$ .  $\tilde{u}_i^\tau$  coincides with  $u_i^\tau$  on  $\square$ , and there exists some  $H^\tau$  such that  $\tilde{u}_i^\tau(0, \dots, 0, H^\tau, 0, \dots, 0) > u_i^*$  for  $H^\tau$  in any component.

Now, if  $\mu^* := \max_i \mu_i$  and  $H := \max_{\tau \in [0,1]} H^\tau$ , we claim that

$$p_\ell^\tau \geq \frac{\mu^*}{H}.$$

Otherwise, any agent  $i$  with  $\mu_i = \mu^*$  could spend her money in order to buy  $H$  units of commodity  $\ell$ , thus obtaining a final utility  $\tilde{u}_i^\tau(0, \dots, 0, H, 0, \dots, 0)$  higher than  $u_i^*$ . A contradiction.  $\square$

### Proof of Theorem 2.2.

We take inspiration from Gale (1976). Suppose that  $(p, r, ((x_i, m_i)))$  and  $(q, r, (y_i, m'_i))$  are two eME of  $(\mathcal{E}^\tau(\mathbf{e}), (\mu_i), M)$  and that  $p \neq q$ . We claim that there must be some proper subset of households  $S \subset \{1, \dots, N\}$  such that, if  $L_S \subset \{1, \dots, L\}$  is the subset of goods held by at least one member of  $S$ :

$$L_S := \{\ell \mid \exists i \in S \mid \mathbf{e}_i^\ell > 0\},$$

then,  $\mathbf{e}_j^\ell = 0$  for all  $j \notin S$  and  $\ell \in L_S$ , and  $\sum_S x_i = \sum_S y_i = \sum_S \mathbf{e}_i$ . In words, members of  $S$  initially own all the goods in  $L_S$  (and only these goods) and they trade only among themselves, be it under  $x$  or under  $y$ .

Suppose, for a moment, that the claim is true. Evidently, a 1-agent economy cannot admit any effective eME. Suppose therefore that  $\mathcal{E}$  contains two households,  $N = 2$ . Then, if at some eME,  $(p, r, (x_i, m_i))$ ,  $\mathcal{E}$  admits some independent subset  $S$ ,  $S$  must be a singleton and  $x_i = \mathbf{e}_i$  for every  $i$  – so that, again, all eME must be autarkic. Thus, no two-agent economy can admit independent subsets at any eME. It follows that (nominal) uniqueness of effective eME obtains for  $N = 2$ . We then proceed by induction on the number,  $N$ , of consumers. Suppose that nominal uniqueness of effective eME obtains for every population of size  $1 \leq k \leq N$ , and consider an economy  $(\mathcal{E}, (\mu_i), M)$  with  $N + 1$  households. Consider some eME,  $(p, r, (x_i, m_i))$ , and suppose that  $S \subset \{1, \dots, N + 1\}$  is an independent subset. For any vector  $x \in (\mathbb{R}_+^L)^N$ , denote by  $x_{|S} := (x_i)_{i \in S}$  its restriction to the sub-population  $S$ . Since  $S$  is independent,  $(p, r, (x_{|S}, m_{|S}))$  must be a eME of the sub-economy obtained by disregarding consumers not in  $S$  and by injecting the adequate quantity,  $M_S = \sum_S m_i = (p/1 + r) \cdot \sum_S (x_i - \mathbf{e}_i)^-$ , of “helicopter money” on the short-loan market. In particular, since the budget constraint of each member of  $S$  is binding,  $p \cdot \sum_S z_i = \sum_S (\mu_i - r m_i)$ . As  $S$  is independent,  $\sum_S z_i = 0$ , so that  $r = \sum_S \mu_i / \sum_S m_i$ . Hence,  $(p, r, (x_{|S}, m_{|S}))$  is a eME of  $(\mathcal{E}_S, \mu_S, M_S)$ , and, by assumption,  $p$  is unique. Since utilities are strictly quasi-concave, uniqueness of the equilibrium allocation follows.

We now prove the claim. Since  $p \neq q$ , they cannot be proportional either. Indeed, suppose there was some  $t > 0$  such that  $p = tq$ . Then the budget constraint (b) in Lemma 2.1 would imply

$$tq \cdot (x_i - \mathbf{e}_i)^+ - \frac{t}{1+r} q \cdot (x_i - \mathbf{e}_i)^- = \mu_i,$$

which means that there exists cash borrowing strategies,  $(\tilde{m}_i)_i$ , such that  $(q, r, (\mathbf{e}_i + \frac{1}{t}(x_i - \mathbf{e}_i)), (\tilde{m}_i))$  is a third  $\tau$ -local eME. But  $q \cdot \frac{1}{t}(x_i - \mathbf{e}_i)^- = \tau(1+r)\tilde{m}_i$ , every  $i$ , implies that  $\tilde{m}_i = \frac{1}{t}m_i$ . And  $\sum_i \tilde{m}_i = \sum_i m_i = M$  leads to  $t = 1$ . Therefore, it suffices to prove that  $p$  and  $q$  are not proportional.

Let  $\lambda := \max_\ell q_\ell/p_\ell$  and  $H := \{\ell : q_\ell/p_\ell = \lambda\}$ . Since  $p$  and  $q$  are not proportional,  $H$  is a proper subset of  $\{1, \dots, L\}$ . If  $\lambda \leq 1$  this means that  $q_h \leq p_h$  for every  $h$ . Hence, by interverting the role of  $p$  and  $q$  if necessary, we can assume that  $\lambda \geq 1$ . Now, let  $S := \{i = 1, \dots, N \mid y_{i\ell} > 0 \text{ for some } \ell \in H\}$ . We first show that, for every  $i \in S$ , if  $x_{ih} > 0$ , then  $h \in H$ . Suppose, indeed, that  $i \in S$  and  $h \notin H$ . Then,  $q_h/p_h < \lambda$ . Since  $y_{i\ell} > 0$  and  $x_{i\ell} > 0$  for some  $\ell \in H$ , one has, applying Lemma 2.2:

- either  $i$  is in Case (a) under  $q$  and

$$\frac{\frac{\partial}{\partial x_i^\ell} u_i^\tau(y_i)}{q_\ell} \geq \frac{\frac{\partial}{\partial x_i^h} u_i^\tau(y_i)}{q_h} > \frac{\frac{\partial}{\partial x_i^h} u_i^\tau(y_i)}{\lambda p_h};$$

- or  $i$  is in Case (b) under  $q$  and

$$\frac{\frac{\partial}{\partial x_i^\ell} u_i^\tau(y_i)}{(1+r)q_\ell} \geq \frac{\frac{\partial}{\partial x_i^h} u_i^\tau(y_i)}{q_h} > \frac{\frac{\partial}{\partial x_i^h} u_i^\tau(y_i)}{\lambda p_h}.$$

In both cases, multiplying by  $\lambda > 0$  yields:

$$\frac{\frac{\partial}{\partial x_i^\ell} u_i^\tau(y_i)}{p_\ell} > \frac{\frac{\partial}{\partial x_i^\ell} u_i^\tau(y_i)}{(1+r)p_\ell} > \frac{\frac{\partial}{\partial x_i^h} u_i^\tau(y_i)}{p_h}. \quad (12)$$

Next, Assumption (D) guarantees that, for each  $k = 1, \dots, L$ , the partial derivative mapping  $x_i \mapsto \frac{\partial}{\partial x_i^k} u_i(\mathbf{e}_i + \tau(x_i - \mathbf{e}_i))$  locally uniformly converges towards the constant  $\frac{\partial}{\partial x_i^k} u_i(\mathbf{e}_i)$  as  $\tau \rightarrow 0^+$ . More precisely, applying the Taylor rule and Prop. 3.1. of Bonnisseau & Orntangar (2010) to  $x_i \mapsto \frac{\partial}{\partial x_i^k} u_i(\mathbf{e}_i + \tau(x_i - \mathbf{e}_i))$  yields the existence of some compact  $K_k^i \subset \mathbb{R}^L$  containing  $\mathbf{e}_i$ , for which, for every  $\varepsilon > 0$ , there exists  $\bar{\tau}$  such that, for all  $\tau \leq \bar{\tau}$ , and all  $x_i \in K_k^i$ ,

$$\left\| \frac{\partial}{\partial x_i^k} u_i(\mathbf{e}_i + \tau(x_i - \mathbf{e}_i)) - \frac{\partial}{\partial x_i^k} u_i(\mathbf{e}_i) \right\| \leq \varepsilon.$$

Thus, for  $\bar{\tau}$  small enough, and thanks to Lemma 2.3, we get:

$$\left\| \frac{\frac{\partial}{\partial x_i^\ell} u_i^\tau(y_i)}{p_\ell} - \frac{\frac{\partial}{\partial x_i^\ell} u_i^\tau(x_i)}{p_\ell} \right\| \leq \frac{2\varepsilon}{\eta},$$

so that (12) translates into the analogous inequalities taken at the base point  $x_i$  (instead of  $y_i$ ):

$$\frac{\frac{\partial}{\partial x_i^\ell} u_i^\tau(x_i)}{p_\ell} > \frac{\frac{\partial}{\partial x_i^\ell} u_i^\tau(x_i)}{(1+r)p_\ell} > \frac{\frac{\partial}{\partial x_i^h} u_i^\tau(x_i)}{p_h}.$$

Applying Lemma (2.2) (ii), we conclude that  $x_{ih} = 0$ .

Suppose, now, with no loss of generality, that  $H =: \{1, \dots, \ell\} \subset \{1, \dots, L\}$ . For any vector  $x \in \mathbb{R}^L$ , write  $x_{|\ell}$  for its truncation  $(x_1, \dots, x_\ell)$ . Since members of  $S$  only receive goods in  $H$  at  $p$ , their whole wealth must be spent on commodities in  $H$ . But since they receive only goods in  $H$ , this means that, at  $p$ , they sell at least their entire endowments of goods not in  $H$ . Let us denote by  $\tilde{\omega}_i$  the vector defined by:

$$\tilde{\omega}_{i\ell} := \begin{cases} \omega_{i\ell} & \text{if } \ell \notin H \\ 0 & \text{otherwise.} \end{cases}$$

One must have:

$$\sum_S \left( \mu_i + \frac{p \cdot \tilde{\omega}_i}{1+r} \right) \leq \sum_{i=1}^N p_{|\ell} \cdot \omega_{|\ell}. \quad (13)$$

On the other hand, the members of  $S$  must be able to afford all the commodities in  $H$  at prices  $q$ . The maximal wealth they can dispose of in order to do this would result from the sale of their entire endowments not in  $H$ . Therefore,

$$\sum_S \left( \mu_i + \frac{q \cdot \tilde{\omega}_i}{1+r} \right) \geq \sum_{i=1}^N q_{|\ell} \cdot \omega_{|\ell} = \lambda p_{|\ell} \cdot \sum_{i=1}^N \omega_{|\ell}. \quad (14)$$

Multiplying (13) by  $\lambda > 0$ , and subtracting it from (14) yields:

$$(1 - \lambda) \sum_S \mu_i + \sum_S \frac{(q - \lambda p) \cdot \tilde{\omega}_i}{1+r} \geq 0.$$

Since  $\lambda \geq 1$ , while  $q_h - \lambda p_h \leq 0$  for every good  $h$ , the inequality being strict for  $h \notin H$ , we get  $\omega_{ih} = 0$  for every  $i \in S$  and every  $h \notin H$ . It follows that  $S$  is independent. □

Theorem 2.2 focuses on effective eME. We now turn to autarkic equilibria. Utilities are said to be separable if  $u_i(x_i) = \sum_\ell u_i^\ell(x_i^\ell)$ . Cobb-Douglas and log-linear utilities are separable.

**Proposition 2.1** *Under (D),*

(i) *when  $\bar{\mu}/M > \gamma(\mathbf{e})$ , no-trade is the unique  $\tau$ -local eME for  $\tau$  sufficiently small.*

(ii) *When  $\bar{\mu}/M \geq \gamma(\mathbf{e})$ , no-trade is a  $\tau$ -local eME for  $\tau$  small enough. Moreover, if utilities are separable and strictly concave, it is the unique  $\tau$ -local eME.*

(iii) *Conversely, under (D) and (GT), autarky cannot be a  $\tau$ -local eME for  $\tau$  sufficiently small.*

**Proof of Proposition 2.1.**

(i) Given some  $\tau > 0$ , consider the subset of allocations that are feasible and individually rational:

$$\mathcal{A}^\tau(x) := \left\{ (y_i) \in \mathbb{R}_+^{N\ell} \mid \sum_i (y_i - \mathbf{e}_i) = 0, y_i \geq (1 - \tau)\mathbf{e}_i \forall i, \right. \\ \left. \text{and } u_i(y_i) \geq u_i(x_i), i = 1, \dots, N \right\}.$$

Define the  $\tau$ -localized upper gains-to trade as:  $\Gamma^\tau(\mathbf{e}) := \sup\{\gamma(y) \mid y \in \mathcal{A}^\tau(\mathbf{e})\}$ . Repeating verbatim the proof of Theorem 6 in Dubey & Geanakoplos (2003a) adapted to  $\tau$ -localized trades (i.e., which verify  $y_i \geq (1 - \tau)\mathbf{e}_i$ ), yields: As soon as  $\frac{\bar{\mu}}{M} > \Gamma^\tau(\mathbf{e})$ , then no effective  $\tau$ -local eME exists.

On the other hand, when  $\tau = 0$ , then  $\mathcal{A}^0(x) = \{\mathbf{e}\}$  and  $\Gamma^0(\mathbf{e}) = \gamma(\mathbf{e})$ . Now, suppose that  $\frac{\bar{\mu}}{M} > \gamma(\mathbf{e})$ . The application  $\tau \mapsto \Gamma^\tau(\mathbf{e})$  is continuous. Indeed, the correspondence  $\tau \Leftrightarrow \mathbf{A}^\tau(x)$  is clearly continuous (given the continuity of utility functions  $u_i(\cdot)$ ), and takes values within the (compact) feasible set. On the other hand,  $\gamma(\cdot)$  is a continuous function (given the differentiability of utilities). Continuity of  $\tau \mapsto \Gamma^\tau(\mathbf{e})$  follows from Berge maximum theorem. It implies, for  $\tau$  sufficiently small:

$$\frac{\bar{\mu}}{M} > \Gamma^\tau(\mathbf{e}) \geq \gamma(\mathbf{e}).$$

Hence, for  $\tau$  sufficiently small, no effective  $\tau$ -local eME exists. Next, we prove that no-trade is a  $\tau$ -local eME. According to the very definition of  $\gamma(\mathbf{e})$ , the second welfare theorem yields a price vector  $p \in \mathbb{R}_+^L$  such that  $(\mathbf{e}, p)$  is a Walrasian equilibrium of  $\mathcal{E}(\mathbf{e})$ . Hence, since  $r > \gamma(\mathbf{e})$ , for every household  $i$  and every pair,  $(k, \ell)$ , of goods, one has:

$$\frac{\frac{\partial}{\partial x_i^\ell} u_i(\mathbf{e}_i) / p_\ell}{\frac{\partial}{\partial x_i^k} u_i(\mathbf{e}_i) / p_k} \leq 1 + \gamma(\mathbf{e}) < 1 + r. \quad (15)$$

For every good  $k = 1, \dots, L$ , the uniform convergence, proven in Lemma 2.2, of the partial derivatives of  $\tau$ -localized utilities,  $\frac{\partial}{\partial x_i^\ell} u_i^\tau(\cdot)$ , towards the  $k$ -coordinate of the gradient,  $\nabla u_i(\mathbf{e}_i)$ , implies that (18) holds for  $\tau$ -localized utilities when  $\tau$  is sufficiently small:

$$\frac{\frac{\partial}{\partial x_i^\ell} u_i^\tau(\mathbf{e}_i) / p_\ell}{\frac{\partial}{\partial x_i^k} u_i^\tau(\mathbf{e}_i) / p_k} < 1 + r. \quad (16)$$

This means that no individual  $i$  has any incentive to sell commodity  $k$  against commodity  $\ell$  — she would improve her welfare by slightly reducing her bank loan, the amount of good  $k$  she sells and curtailing her demand for good  $\ell$ . Thus, if the prevailing price system is  $p$ , the solution of each household's  $i$  program of  $\tau$ -local utility-maximization yields  $z_i^{*-} = m_i^* = 0$ , together with some  $z_i^{*+}$  such that  $p \cdot z_i^{*+} = \mu_i$ . According to condition (ii) in the definition of an eME, it follows that  $x_i = \mathbf{e}_i$ , turns this action profile into an (autarkic)  $\tau$ -local eME.

(ii) If  $\frac{\bar{\mu}}{M} \geq \gamma(\mathbf{e})$ , the proof of (i) holds as well, proving that no-trade is a  $\tau$ -local eME for  $\tau$  small enough. On the other hand, when utilities are separable and strictly concave, Theorem 7 in Dubey & Geanakoplos (2003a) shows that an effective ME exists if, and only if,  $\frac{\bar{\mu}}{M} > \gamma(\mathbf{e})$ . Therefore, when  $\frac{\bar{\mu}}{M} = \gamma(\mathbf{e})$ , and utilities are separable and strictly concave, then no-trade is the unique  $\tau$ -local eME for  $\tau$  small enough.

(iii) Suppose that (GT) and (D) hold, and that no-trade is a  $\tau$ -local eME, however small  $\tau$  may be. Consider a sequence  $\tau(k) \rightarrow 0^+$ . For every integer  $k$ , one must have, for every  $(i, k, \ell)$ :

$$\frac{\frac{\partial}{\partial x_i^\ell} u_i^{\tau(k)}(\mathbf{e}_i) / p_\ell}{\frac{\partial}{\partial x_i^k} u_i^{\tau(k)}(\mathbf{e}_i) / p_k} < 1 + r. \quad (17)$$

Otherwise, agent  $i$  would have an incentive to increase her demand for commodity  $k$  and her offer of commodity  $\ell$ . Taking the limit yields:

$$\frac{\frac{\partial}{\partial x_i^\ell} u_i(\mathbf{e}_i) / p_\ell}{\frac{\partial}{\partial x_i^k} u_i(\mathbf{e}_i) / p_k} \leq 1 + r. \quad (18)$$

This being true for every triple  $(i, k, \ell)$ , we can consider any cycle of trades among households — a cycle,  $c_n$ , is a sequence of distinct commodities  $(\ell_1, \dots, \ell_n)$  and agents  $(i_1, \dots, i_n)$  where  $i_k$  buys  $\ell_k$  and sells  $\ell_{k+1}$  (with  $\ell_{n+1} := \ell_1$ ). For such a cycle,  $c_n$ , one has:

$$\prod_{k=1}^n \left[ \frac{\partial}{\partial x_{i_k}^{\ell_{k+1}}} u_{i_k}(\mathbf{e}_{i_k}) / \frac{\partial}{\partial x_{i_k}^{\ell_k}} u_{i_k}(\mathbf{e}_{i_k}) \right] \leq (1 + r)^{\frac{1}{n}}.$$

Taking the maximum in the previous inequality among the (finite) set of cycles leads to:

$$(1 + \gamma(\mathbf{e}))^{\frac{1}{n}} \leq (1 + r)^{\frac{1}{n}}.$$

A contradiction with (GT). Thus, there exists some  $\bar{\tau}$  small enough, so that, for every  $\tau \leq \bar{\tau}$ , no-trade is not a  $\tau$ -local eME. □

Thus, under (D), given initial endowments  $\mathbf{e}$ , the situation is as follows (for  $\tau$  sufficiently small):

- If  $r < \gamma(\mathbf{e})$ , there exists a unique  $\tau$ -local eME, which is effective.
- If  $r = \gamma(\mathbf{e})$ , no-trade is a  $\tau$ -local eME. It is the unique one if utilities are strictly concave and separable.
- If  $\gamma(\mathbf{e}) < r$ , no-trade is the unique  $\tau$ -local eME.

### 3 Monetary policy under myopia

We are now ready to describe the trading process, and to examine the impact of households' myopia on monetary policy.

From now on, we fix a monetary policy,  $(M(t))_{t=1,\dots,T}$ , of the Bank, consisting in injecting  $M(t) \geq 0$  on the short-term loan market, at each period  $t = 1, \dots, T$ . A spot interest rate,  $r(t) \geq 0$ , forms, so as to clear the current short-term loan market.

The profit of the Bank at time  $t$  on the short term loan market is:

$$\pi(t) := r(t)M(t).$$

At the beginning of period  $t + 1$ , household  $i$  receives a dividend income corresponding to a portion,  $\theta_i \pi(t)$ , of the time  $t$ -profit of the Bank, where  $\theta_i \geq 0$  is an exogeneous share such that  $\sum_i \theta_i = 1$ .<sup>16</sup>

Hence, for  $t \leq T - 1$ , the cash balances of individual  $i$  at the beginning of  $t + 1$  are given by:

$$\mu_i(t + 1) := \mu_i(t) + \theta_i \pi(t) - p(t) \cdot z_i(t) - r(t)m_i(t) \quad (19)$$

where  $z_i(t) := x_i(t) - x_i(t - 1)$  is the exchange vector. Thus,  $\mu_i(1)$  is some outside money, and  $\mu_i(t)$  is the amount of cash available to agent  $i$  at the beginning of period  $t$  given the history of her trading strategies. Equation (19) provides the dynamics of  $\mu_i(\cdot)$  for each agent  $i$ .

#### 3.1 A discrete trading process

Let

$$ME : \mathbf{E} \times [0; \hat{\mu}] \times \mathbb{R} \longrightarrow \mathbb{R}_{++}^L \times \mathbb{R}_+ \times (\mathbb{R}_+^L)^N$$

be the short-term equilibrium map which associates to each collection  $(\mathbf{e}, (\mu_i), M)$  the unique  $\tau$ -local monetary equilibrium (cf. Theorem 1.1. supra). Let  $\hat{M}E$  denote the canonical projection of  $ME$  on the price and commodities spaces, that is,  $\hat{M}E(\mathbf{e}, (\mu_i), M) = (p, x)$  is the couple of equilibrium price and allocation.

**Definition 3.1** A *monetary exchange process* controlled by  $(\tau(t))_t$  and the monetary policy  $M(t)_t$  is a sequence  $(p(t), r(t), x(t), \mu(t))$  in  $\mathbb{R}_+^{L+1} \times \mathbf{E} \times [0, \hat{\mu}]^N \times \mathbb{R}$ , such that (19) holds and:

$$(p(t + 1), x(t + 1)) = \hat{M}E(x(t), \mu(t), M(t)) \quad t = 1, \dots, T - 1, \quad (20)$$

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<sup>16</sup>In the United States, individuals own private banks, which in turn have shares in the Federal Reserve. But the Federal Reserve does not distribute all its revenue to these banks. Routinely the Fed turns over money to the Treasury. We could easily capture this feature by setting  $0 < \sum_i \theta_i < 1$ . This would induce the automatic reduction of outside money at each period by the amount  $1 - \sum_i \theta_i$ . In posing  $\sum_i \theta_i = 1$ , we follow Drèze & Polemarchakis (2005).

with  $x(0) := \mathbf{e}$  and  $p(0) \in \mathbb{R}_+^L$ .

**Remark 3.1** At time  $t$ , the  $\tau$ -local eME verifies:  $\mu_i(t) = p(t)z_i(t) + r(t)m_i(t)$ . Hence, (19) can be rewritten as

$$\mu_i(t+1) := \theta_i \pi(t) \quad (21)$$

Along an exchange process, (21) is supposed to hold at every period  $t$ : Default is not allowed in this paper. Since  $\pi(t) = r(t)M(t) + \frac{\bar{r}}{T}M(t) = \bar{\mu}(t) + \frac{\bar{r}}{T}M(t)$  along any exchange process, (21) implies:

$$\mu_i(t+1) = \bar{\mu}(t) \frac{\mu_i(1)}{\bar{M}} \quad \forall t \leq T-1, \text{ every } i.$$

It follows that  $\bar{\mu}(t+1) = \bar{\mu}(t) = \bar{M}$  for every  $1 \leq T-1$ . In other words, the quantity of money in the households' hands at the beginning of each period  $t \leq T$  remains constant. Finally, we need a standard boundary condition:

### Assumption (B)

a) (*Boundary behavior*) for all  $\underline{x}_i \in \mathbb{R}_{++}^L$ , the set

$$U_i(\underline{x}_i) = \{x_i \in \mathbb{R}_{++}^L \mid u_i(x_i) \geq u_i(\underline{x}_i)\} \text{ is a closed subset of } \mathbb{R}^L;$$

b) (*Survival condition*)  $\mathbf{e}_i \in \mathbb{R}_{++}^L$ .

**Theorem 3.1** (i) Under (C) and (P), the monetary economy  $\mathcal{E}$  admits a monetary exchange process  $(p(t), r(t), x(t), \mu(t))$ , whatever being the monetary policy  $(M(t))$ , provided  $\tau$  is small enough.

Suppose, in addition, that (D) and (B) are in force, and that (GT) holds at each period  $0 \leq t \leq T$ .

(ii) For every  $\varepsilon > 0$ , there exists  $\bar{t} \in \mathbb{N}$  such that, if  $T \geq \bar{t}$ ,

$$\gamma(x(T)) \leq \varepsilon,$$

(iii) The monetary exchange process,  $(p(t), r(t), x(t), \mu(t))$ , is unique. Moreover, the price sequence  $(p(t))$  converges to the supporting price of the unique (Pareto-optimal) limit-point,  $x^* := \lim_{t \rightarrow \infty} x(t)$ .

(iv) If, on the contrary,  $r > \gamma(x(t^*))$  for some date  $t^* < T$ , then,  $x(t) = x(t+1)$  for every  $t^* \leq t \leq T$  for which  $r > \gamma(x(t))$ .



**Remark 3.2** Given some admissible monetary path, if the number of rounds is sufficiently large, then trades will asymptotically lead to a Pareto-optimal allocation. And the approximation can be made arbitrarily coarse by adding sufficiently many rounds of trades. This contrasts with Bonnisseau & Orntangar (2010) where a finite number of rounds was sufficient to get exact efficiency. The presence of a limited quantity of money is responsible for convergence not to be in finite time in the present paper: In a sense, the Arrow-Debreu model depicts an idealized world where an infinite amount of money is freely available to all traders. If one gives up this unrealistic assumption, then, households who borrow money lose the interest-float on their marginal purchases which discourages some trades. The upshot is that an effective monetary equilibrium allocation (whether  $\tau$ -local or not) cannot be Pareto-optimal, always leaving room for further gains to trade.

The last result also provides us with a definition of a minimal growth rate of money. Indeed, if  $M(t)$  grows sufficiently rapidly for (GT) to remain in force along the whole trading path, then the economy will converge towards a unique, efficient rest-point. On the contrary, if  $M(t)$  grows too slowly, so that  $\gamma((t^*)) < \bar{\mu}/M$  at some time  $t^*$ , then the economy remains stuck in some inefficient rest-point until the Bank injects sufficiently money in the system so as to ensure that

$$\frac{\bar{\mu}}{M} < \gamma((t^*)).$$

We interpret this situation as a “local liquidity trap”. Indeed, if the government pumps in more Bank money into the economy, but not sufficiently many, so that  $\gamma((t^*)) < \bar{\mu}/M$  still prevails, then it does not succeed in convincing agents to trade. However, there is a level of money injection that will enable the economy to start trading again. This is the sense in which a liquidity trap, in this paper, is only local. We see this phenomenon as a theoretical ground for the scenario observed during the recent financial crisis.<sup>17</sup>

**Remark 3.3** Another consequence of this Theorem is that money is non-neutral, neither in the short-, nor in the long-run. Consider, indeed, two distinct monetary policies,  $(M(t)), (M'(t))$ , such that, say,  $M(t) = M$  and  $M'(t) = M'$  for every  $t$ , with  $M > M' > \bar{\mu}/\gamma(\mathbf{e})$ . For both policies, the economy will end up in a local liquidity trap (for  $T$  sufficiently large). But, in general, trades will cease at different times when  $(M'(t))$  or  $(M(t))$  are put into practice.

Let us prepare the proof of Theorem 3.1 by the following lemma – a discrete-time and set-valued version of Lyapunov’s second method.

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<sup>17</sup>Here, for simplicity, we assumed that every agent is purely myopic. In a companion paper, we show how the phenomenon of local liquidity traps survives even if households share rational expectations.

**Lemma 3.1** *Let  $U \subset \mathbb{R}^n$  be a compact subset, and  $F : \mathbb{R}^n \rightarrow 2^U$  a set-valued map with a closed graph. Suppose*

$$\forall k, x(k+1) \in F(x(k)) \text{ and } x(0) \in U.$$

*If there exists a continuous function  $\mathcal{L} : U \rightarrow \mathbb{R}$  and a non-empty subset  $P \subset U$  such that:*

- *if  $x \in U \setminus P$ ,  $\mathcal{L}(y) < \mathcal{L}(x) \forall y \in F(x)$ ;*
- *if  $x \in P$ ,  $F(x) = \{x\}$ .*

*then,  $\forall \varepsilon > 0, \exists k \in \mathbb{N} : k' > k, d(x(k), P) < \varepsilon$ .*

### Proof of Lemma 3.1

First, if there exists some  $k$  such that  $x(k) \in P$ , then  $x(k+1) = x(k)$ , and we are done. Suppose, on the contrary, that  $x(k) \notin P$  for all  $k$ . By compactness of  $U$ , there exists a subsequence  $\phi(\cdot) : \mathbb{N} \rightarrow \mathbb{N}$  such that the sequence  $(x(\phi(k)))$  converges towards  $z \in U$ . Suppose that  $z \notin P$ . Then,  $\mathcal{L}(y) < \mathcal{L}(z)$  for every  $y \in F(z)$ . By continuity of  $\mathcal{L}$  and the graph-closedness of  $F$ , there exists a neighborhood  $V$  of  $z$  such that  $\mathcal{L}(F(y)) < \mathcal{L}(z)$  for every  $y \in V$ . On the other hand, there exists some integer  $k_0$  for which  $\phi(x(k_0)) \in V$ , hence,  $\mathcal{L}(F(\phi(x(k_0)))) < \mathcal{L}(z)$  (i.e., the inequality is true for every  $y \in F(\phi(x(k_0)))$ ). So that

$$\mathcal{L}(F(\phi(x(k_0+1)))) < \mathcal{L}(F(\phi(x(k_0)))) < \mathcal{L}(z).$$

This contradicts the fact that  $(x(\phi(k)))_k$  converges towards  $z$ . Therefore,  $z \in P$ . Now,  $P$  being compact, so is its (closed) tubular neighborhood  $\mathcal{N}(P, \varepsilon) := \{y \in U \mid d(y, P) \leq \varepsilon\}$ . The continuous function  $\mathcal{L}$  reaches a maximum, say  $\eta$ , over  $\mathcal{N}(P, \varepsilon)$ .  $\mathcal{L}(\cdot)$  being decreasing, there exists some  $k'$  such that:  $\forall k \geq k', \mathcal{L}(x(k)) < \eta$ .

□

### Proof of Theorem 3.1

(i) is a direct consequence of earlier results: For each  $0 \leq t \leq T$ , either  $\frac{\mu}{M} \geq \gamma(x(t))$  — in which case, no-trade is an eME (Proposition 2.1 (ii))—, or  $\gamma(x(t)) > \frac{\mu}{M}$ , and there exists some effective ME (Theorem 2.1 (ii)).

(ii) It suffices to take the feasible set for  $U$ ,

$$U := \{(x_i)_i \in (\mathbb{R}_+^L)^N \mid \sum_i (x_i - \omega_i) = 0\},$$

the Pareto set for  $P$ , and  $\mathcal{L}(t) := -\sum_i u_i(x_i(t))$  as Lyapunov function in order to get the desired convergence result. Indeed, the individual rationality of each eME implies that  $\mathcal{L}(\cdot)$  is non-decreasing along a trade path. Moreover, the set-valued map  $\hat{M}E(\cdot)$ <sup>18</sup> clearly has a closed graph. Finally, under (GT) and (C), we claim that the two following assertions are equivalent:

<sup>18</sup>Observe that, even under (D) and (B), we cannot guarantee the uniqueness of the eME at  $x$ , whenever  $\bar{\mu}/M = \gamma(x)$  — unless utilities are separable.

a) the  $\tau$ -local eME of  $(\mathcal{E}(x(t-1)), (\mu_i(t)), M(t))$ , verifies:

$$x_i(t) = x_i(t-1) \quad \forall i;$$

b)  $x(t-1)$  is Pareto-optimal (i.e.,  $x(t-1) \in P$ ).

Suppose that the  $\tau$ -local eME,  $((x_i(t), m_i(t)), p(t), r(t))$ , is effective, and yet yields the same final utility level to each household  $i$  as would the consumption of one's initial endowment:

$$u_i(x_i(t)) = u_i(x_i(t-1)) \quad i = 1, \dots, N.$$

By the strict quasi-concavity of preferences, any convex linear combination  $y_i := \lambda x_i(t) + (1-\lambda)x_i(t-1)$  with  $\lambda \in (0, 1)$  would be strictly preferred by every  $i$  to  $x_i(t)$ . Since  $(y_i)$  would be feasible and compatible with all the individual budget constraints required for a  $\tau$ -local eME, this contradicts the equilibrium character of  $((x_i(t), m_i(t)), p(t), r(t))$ . Hence, if  $((x_i(t), m_i(t)), p(t), r(t))$  is not autarkic, it must strictly improve the individual welfare of at least one household  $i$ . Now, if  $\gamma(x(t)) > \frac{\bar{\mu}}{M}$ , we know from Theorem 2.2 that the unique  $\tau$ -local eME is effective (for  $\tau$  small enough). This proves a)  $\Rightarrow$  b).

If  $\gamma(x(t)) = 0$ , then  $\frac{\bar{\mu}}{M} \geq \gamma(x(t))$ , and Proposition 2.2 implies that no-trade is a eME. By individual rationality of eME, there is no other eME. Hence, b)  $\Rightarrow$  a).

(iii) We claim that, at a  $\tau$ -local eME,  $((x_i(t), m_i(t)), p(t), r(t))$ , any active household  $i$  borrowing money, purchasing good  $k$  and selling only a part of her endowment  $x_i(t-1)$  of commodity  $\ell$  (thanks to the boundary condition and for  $\tau$  sufficiently small), verifies:

$$\frac{\frac{\partial u_i}{\partial x_i^\ell}(x_i(t))}{p_\ell(t)} = (1 + r(t)) \frac{\frac{\partial u_i}{\partial x_i^k}(x_i(t))}{p_k(t)}. \quad (22)$$

Suppose, on the contrary, that  $\frac{\partial u_i}{\partial x_i^\ell}(x_i(t))/p_\ell(t) > (1 + r(t)) \frac{\partial u_i}{\partial x_i^k}(x_i(t))/p_k(t)$ . Then,  $i$  could improve its utility by borrowing  $\delta > 0$  additional Euros from the Bank and spending them to purchase good  $\ell$ , while defraying the loan by selling  $(1 + r(t))\delta$  Euros' worth more of good  $k$ . If the reverse inequality holds,  $i$  would benefit by reducing slightly its bank loan and purchasing of  $\ell$  while curtailing the concomitant sale of  $k$ .

On the other hand, it follows from parts (i) and (ii) of Theorem 3.1., that there exists some  $t^*$  such that, from  $t^*$  on, global gains to trade are decreasing that is :  $\gamma(x(t+1)) \leq \gamma(x(t))$ , every  $t \geq t^*$ . Hence, (GT) implies that  $r(t) \rightarrow 0^+$  along a trade path, so that the price vector  $p(t)$  becomes asymptotically collinear to the gradient of active households as  $t \rightarrow \infty$ . Since  $x(t)$  converges towards a Pareto-optimal point (where all the household's gradients are colinear to the price vector sustaining this very Pareto point as a no-trade Arrow-Debreu equilibrium), the conclusion follows.

(iv) is a direct consequence of Proposition 2.1. □

### 3.2 Taylor rule revisited

An important consequence of the last Theorem deals with the Taylor rule. Following Taylor (1993), we describe this “rule” as specifying how much the Central Bank should change the nominal short-term interest rate,  $r$ , in response to divergences of actual inflation rates from target inflation rates and of actual Gross Domestic Product (GDP) from potential GDP. The rule can be written as follows:

$$r(t) = \pi(t) + r^*(t) + a(\pi(t) - \bar{\pi}(t)) + b(y(t) - \bar{y}(t)), \quad (23)$$

where  $a, b \geq 0$ ,  $\pi(t)$  is the rate of inflation as measured by the GDP deflator,  $r^*(t)$  is some target real interest rate,  $y(t)$  is the logarithm of real GDP and  $\bar{y}(t)$  is the logarithm of some potential output (as determined, say, by a linear trend or any alternate rule of thumb). The GDP deflator is usually understood as the ratio between nominal and real GDP. How should we define it in our set-up ? Following Dubey & Geanakoplos (2003a), we consider the value,  $p(t) \cdot \sum_i z_i^+(t)$ , of aggregate expenditures at time  $t$  as playing the role, in our setting with heterogeneous households and multiple commodities, of nominal GDP. Real GDP could be approximated by the “size” of these expenditures, say, the norm  $\|\sum_i z_i^+(t)\|$ . Thus, we understand (23) with the following interpretation:

$$\pi(t) := \frac{p(t) \cdot \sum_i z_i^+(t)}{\|\sum_i z_i^+(t)\|} \quad \text{and} \quad y(t) := \ln \|\sum_i z_i^+(t)\|.$$

**Corollary 3.1** *Under (D), (B) and (P), for any monetary policy rule obeying (23) (i.e., for any choice of the parameters  $a, b$  and the targets  $r^*(t)$  and  $\bar{y}(t)$ ), there exists some  $T$  sufficiently large so that the economy remains stuck at a local liquidity trap at some finite time  $t < T$ .*

#### Proof of Corollary 3.1

Suppose that the Central Bank follows (23), and assume that the economy never stops at some autarkic rest-point until  $T$ . According to Theorem 3.1 (i), the trade path converges to some Pareto-point as  $T \rightarrow \infty$ . Hence  $\gamma(x(t)) \rightarrow 0^+$ . For this, it must be the case, according to Theorem 3.1 (iii), that:  $M(t) > \bar{\mu}/\gamma(x(t))$  for each  $t$ . Hence,  $M(t) \rightarrow +\infty$  as  $t \rightarrow \infty$ . The Quantity Theory of Money (8) then tells us that  $p(t) \cdot \sum_i z_i^+(t) \rightarrow \infty$ . Since trades must be feasible along a solution path,  $\|\sum_i z_i^+(t)\|$  is uniformly bounded. Therefore, the ratio  $\pi(t) \rightarrow +\infty$ . Multiply  $r(t)$  by  $z(t) := \|\sum_i z_i^+(t)\|$ :

$$r(t)z(t) = \pi(t) \left[ 1 + \frac{r^*(t)z(t)}{\pi(t)} + az(t) \left( 1 - \frac{\bar{\pi}(t)}{\pi(t)} \right) + b \left( \frac{z(t)y(t)}{\pi(t)} - \frac{\bar{y}(t)z(t)}{\pi(t)} \right) \right].$$

Clearly,  $\frac{z(t)y(t)}{\pi(t)} = \frac{z(t) \ln z(t)}{\pi(t)} \rightarrow 0^+$  as  $t \rightarrow +\infty$ . Thus,  $r(t)z(t) \rightarrow +\infty$ , which proves that  $r(t) \rightarrow +\infty$ . A contradiction with the fact that Theorem 3.1 (iii) also implies  $\gamma(x(t)) > r(t) \rightarrow 0^+$ . Thus, following (23) must lead to some inefficient autarkic rest-point at some finite  $t < T$ .  $\square$

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## 4 Appendix

### 4.1 Linear monetary economies

In this section, we provide some background material on monetary linear exchange economies that may be of interest in its own right. The results to follow also play a pivotal role for the proof of Theorem 1.1.

Let us therefore consider a linear exchange economy  $\mathcal{L}$  with  $N$  consumers,  $L$  commodities. Each consumer  $i$  is characterized by her utility function :  $u_i : \mathbb{R}_+^L \rightarrow \mathbb{R}$ , defined by  $u_i(x_i) := a_i \cdot x_i$  for some vector  $a_i \in \mathbb{R}_+^L \setminus \{0\}$ , and by her initial endowment,  $\omega_i \in \mathbb{R}_+^L \setminus \{0\}$ . In the following, we assume with no loss of generality that, for every commodity  $\ell$ , there exists  $i$  with  $a_{i\ell} > 0$  and  $j$  with  $\omega_{j\ell} > 0$ .

Each household  $i$  is also endowed with some outside money  $\mu_i \geq 0$ . Again, there is no loss in postulating that  $\bar{\mu} := \sum_i \mu_i > 0$ . Some short-run interest rate  $r > 0$  is fixed by the Central Bank, together with the total amount,  $M > 0$ , of inside money. Given some final allocation,  $(x_i^*)_i$ , the corresponding net trades are written:  $z_i^* := x_i^* - \omega_i$ , every  $i$ . In this Appendix, a **monetary equilibrium** (ME) of  $(\mathcal{L}, \mu, M)$  is then a triple  $((x_i^*)_i, p^*, r^*)$  such that the three following conditions hold:

- (a) For every  $i$ ,  $x_i^*$  maximizes  $a_i \cdot x_i$  over the monetary budget set  $\{x_i \geq 0 \mid p^* \cdot z_i^{*+} + \frac{1}{1+r^*} p^* \cdot z_i^{*-} \leq \mu_i\}$ ;
- (b)  $\sum_i z_i^* \leq 0$ ,
- (c)  $\sum_i \tilde{m}_i(p^*, r^*) = M$ .

Exactly as in the concave situation examined in the body of this paper, we need the gains-to-trade hypothesis borrowed from Dubey & Geanakoplos (2003a) in order to ensure that money will have positive value at equilibrium.

GAINS-TO-TRADE assumption (GT).–  $\gamma(\omega) > \frac{\bar{\mu}}{M}$ .

Observe that, if  $((x_i^*)_i, p^*, r^*)$  is a ME, then  $p^* \gg 0$ . As is well-known, even in the barter case, the above mentioned assumptions on the real part of the economy  $\mathcal{L}$  are not sufficient to guarantee that the set of Walras equilibria is non-empty. Neither are they for the existence of a ME. Consider, for instance, the two-agent, two-good economy  $\mathcal{L}$ , with  $a_1 = (1, 0)$ ,  $a_2 = (0, 1)$ ,  $\omega_1 = (1, 0)$ ,  $\omega_2 = (2, 3)$ ,  $\mu_1 = 1$ ,  $\mu_2 = 0$ , and  $+\infty > r, M > 0$ . It is readily seen that no  $p \gg 0$  can be a ME price vector, since household's 2 demand would exit the attainable set,  $A$ . Neither can  $(0, p_2)$  be a ME price since, then, household's 1 maximization programme has no solution. Observe that, here, non-existence does not arise because of any lack of gains-to-trade since those are clearly positive in  $\mathcal{L}$ , while  $M$  can be chosen arbitrarily large (and actually,  $\bar{\mu}$  arbitrarily small). On the other hand, providing household 2 with some additional outside money would'nt help to restore existence. Therefore, we need some additional restriction akin to the irreducibility requirement in the barter case. If  $\omega_i^\ell > 0$ , say that “agent  $i$  has  $\ell$ ”; if the restriction of  $u_i(\cdot)$  to the projection of the feasible and individually rational set to  $i$ 's consumption set,  $\mathbb{R}_+^L$ , is strictly increasing with respect

to commodity  $\ell$ , say that “agent  $i$  wants  $\ell$ ”. Consider the directed graph  $\Gamma \subset \{1, \dots, L\} \times \{1, \dots, L\}$  on the node-set of commodities, with arc  $(\ell, k)$  if there exists at least one agent  $i$  who has  $\ell$  and wants  $k$ .

HAVING-WANTING CHAIN hypothesis (HW).— For every pair of commodities  $(\ell, k)$  with  $\ell \neq k$ , there exists a directed path from  $\ell$  to  $k$  in  $\Gamma$ .

This assumption stems from the first remark after the proof of Theorem 2 in Dubey & Geanakoplos (2003a), and is slightly stronger than the following monetary adaptation of the standard irreducibility requirement. Assumption (HW) is verified, e.g., either when  $a_i \gg 0$  for every  $i$ , or when  $\omega_i \gg 0$  for every  $i$ .

IRREDUCIBILITY assumption (I).— There do not exist proper subsets  $I$  of the set of households  $\{1, \dots, N\}$  and  $H$  of the set of goods  $\{1, \dots, L\}$ , such that  $\omega_{i\ell} = 0$  whenever  $(i, h) \in I \times H$  and either  $a_{i\ell} = 0$  or  $\mu_i = 0$  whenever  $i \notin I$  and  $\ell \notin H$ .

When  $\mu_i = 0$  for every  $i$ , monetary irreducibility reduces to the standard, barter one. The next two subsections are devoted to proving the following:

**Proposition 4.1** *Under (HW) and (GT):*

- (i) *There exists a ME of the economy  $(\mathcal{L}, \mu, M)$  (Dubey & Geanakoplos (2003a), Theorem 2).*
- (ii) *The set, ME, of monetary equilibria is homeomorphic to a convex set (hence, it is contractible, among other things). The same holds for the set,  $ME_\pi$ , of monetary equilibrium prices. Moreover, if  $(x, p, r)$  and  $(y, q, r)$  are two ME, then, for every  $\alpha \in [0, 1]$ ,  $(\alpha x + (1 - \alpha)y, p^\alpha q^{1-\alpha}, r)$  is also a ME of  $(\mathcal{L}, \mu, M)$ .*
- (iii) *The ME price vector is uniquely determined. That is, if  $((x_i^*)_i, p^*, r^*)$  and  $((y_i^*)_i, q^*, r^*)$  are two ME, then  $p^* = q^*$ .*
- (iv) *The monetary equilibrium utility level of each household  $i$  is uniquely defined. That is, if  $((x_i^*)_i, p^*, r^*)$  and  $((y_i^*)_i, p^*, r^*)$  are two ME, then  $u_i(x_i^*) = u_i(y_i^*)$  for every  $i$ .*

While the restriction from weakly quasi-concave preferences to linear utilities enables to weaken the local non-satiation requirement to mere irreducibility, in part (i) of the present Theorem (as well as in Theorem 2 of Dubey & Geanakoplos (2003a)), the stronger “having-wanting chain” hypothesis is needed in order to ensure that money has a positive value at equilibrium. Parts (ii) and (iii) are the monetary counterparts of Gale’s



(1976) and Cornet's (1989) uniqueness result. Notice, however, that, here, thanks to the presence of money, uniqueness of the *absolute* level of prices holds, and not just that of relative prices as in the barter situation. For the same reason, uniqueness obtains under weaker assumptions than in the barter case: We do not need the additional requirement that there exist at least one household  $i$  with  $\omega_i \gg 0$ , as is imposed in Cornet (1989). Though it cannot be dispensed with in the barter case, this last assumption is superfluous in the monetary case, as we shall see.

## 4.2 Geometric insight

*Proof of part (ii).*

In the following, we denote by  $v_i : \mathbb{R}_+^{L+1} \rightarrow \mathbb{R}$  the indirect utility function of household  $i$ , defined, for every price vector  $p$  and interest rate  $r$ , by:

$$v_i(p, r) := \max \{ a_i \cdot x_i \mid x_i \geq 0, p^* \cdot z_i^{*+} + \frac{1}{1+r} p^* \cdot z_i^{*-} \leq \mu_i \}.$$

We also denote by  $A$  the subset of attainable allocations in  $\mathcal{L}$ :

$$A := \{ (x_i)_i \in \mathbb{R}_+^{LN} \mid \sum_i x_i \leq \sum_i \omega_i \}.$$

*Step 1. Characterization of indirect utilities.*

We begin with a characterization of  $v_i(\cdot, \cdot)$  that will prove useful later on. Given  $(p, r)$ , let us consider the following subset of goods:

$$L^i(p, r) := \left\{ \ell = 1, \dots, L \mid \frac{a_{i\ell}}{(1+r)p_\ell} \geq \frac{a_{ih}}{p_h} \forall h \neq \ell \right\}.$$

Intuitively,  $L^i(p, r)$  is the subset of goods  $\ell$  that are so valuable according to  $i$ 's preferences that she is ready to sell every other commodity  $h \notin L^i$  in order to buy some additional amount of  $\ell$  despite the bid-ask spread introduced by the interest rate,  $r$ .

Two cases have to be distinguished.

**Case A :**  $L^i(p, r) \neq \emptyset$ . Define  $\hat{\omega}_i(p, r)$  as follows, for every good  $\ell$ :

$$\hat{\omega}_{i\ell}(p, r) := \begin{cases} \omega_{i\ell} & \text{if } \ell \notin L^i(p, r), \\ 0 & \text{otherwise.} \end{cases}$$

Given  $(p, r)$ ,  $i$  will but sale the vector  $\hat{\omega}_i(p, r)$  out of her initial endowments. As for commodities in  $L^i(p, r)$ ,  $i$  will keep her initial endowment

and buy as much as possible of these additional goods out of the cash provided by  $\mu_i$  and by her sales  $\frac{1}{1+r}p \cdot \hat{\omega}_i(p, r)$ . Therefore, her utility maximization programme (a) can be rewritten as:

$$\max\{a_i \cdot w \mid p \cdot w \leq \mu_i + \frac{1}{1+r}p \cdot \hat{\omega}_i(p, r) \text{ and } w \geq 0\}.$$

It follows from the duality theorem of linear programming (e.g., Gale (1960)) that  $i$ 's indirect utility function can be written as:

$$v_i(p, r) = \hat{v}_i(p, r) + a_i \cdot [\omega_i - \hat{\omega}_i(p, r)], \quad (24)$$

with

$$\hat{v}_i(p, r) := \left( \mu_i + \frac{1}{1+r}p \cdot \hat{\omega}_i(p, r) \right) \max\{a_{i\ell}/p_\ell \mid \ell \in L^i(p, r)\}. \quad (25)$$

**Case B.** If, now,  $L^i(p, r) = \emptyset$ , there is no good that  $i$  is willing to sell out of her initial endowments. Therefore the unique utility-improving transaction that  $i$  can afford consists in buying additional commodities thanks to her outside money,  $\mu_i$ . Her maximization programme (a) is then:

$$\max\{a_i \cdot w \mid p \cdot w \leq \mu_i \text{ and } w \geq 0\}.$$

So that the duality theorem, again, yields:

$$v_i(p, r) = \tilde{v}_i(p, r) + a_i \cdot \omega_i, \quad (26)$$

with

$$\tilde{v}_i(p, r) := \mu_i \max\{a_{i\ell}/p_\ell \mid \ell = 1, \dots, L\}.$$

The key insight is the following: For each triple  $(x, p, r)$  such that  $a_i \cdot x_i = v_i(p, r)$  ( $i = 1, \dots, N$ ), the set of households can be partitioned into two subsets  $\{1, \dots, N\} = A \cup B$ , where agents  $i \in A$  are in Case A (defined supra) and each  $i \in B$  is in Case B. Consumers in  $B$  do not sell any item: They just buy commodities out of their outside money. On the other hand, since every household  $i \in A \cup B$  fulfills her budget constraint,  $p \cdot z_i^* - \frac{1}{1+r}p \cdot z_i^- \leq \mu_i$ , it follows (e.g., from Lemma 1 in Dubey & Geanakoplos (2003a)) that there exist  $\tilde{m}_i \geq 0$  ( $i = 1, \dots, N$ ) such that:  $p \cdot z_i^* \leq \mu_i + \tilde{m}_i$  ( $i = 1, \dots, N$ ) and  $\sum_i \tilde{m}_i = M$ . Consequently,  $p \cdot \sum_i z_i^+ \leq \bar{\mu} + M$  for every  $i$ . But, when it is binding, the budget constraint also implies that  $p \cdot z_i \leq \mu_i - r\tilde{m}_i$ , for every  $i$ . Therefore,

$p \cdot \sum_i z_i \leq 0$  whenever  $r = \bar{\mu}/M$ . But we have already seen in the body of this paper that this last equality must hold at any ME. It follows that:

$$p \cdot \sum_{i \in A} \hat{\omega}_i(p, r) = p \cdot \sum_i z_i^- \leq \bar{\mu} + M. \quad (27)$$

*Step 2. An equivalent programming problem.*

With this characterization of  $i$ 's indirect utility function in hand, let us prove part (ii) of Proposition 5.1. We partly follow Cornet (1989) taking into account the twist due to the introduction of fiat money. Let consider the following maximization problem in which  $a_i$  and  $\omega_i$  are fixed parameters and  $((x_i)_i, p, r) \in \mathbb{R}^{L(N+1)+1}$  is the variable of the problem:

$$\begin{aligned} (\mathcal{P}) : & \max \min_{i=1, \dots, N} [a_i \cdot x_i - v_i(p, r)] \\ \text{subject to:} & \\ & \sum_i z_i \leq 0, \quad x_i \geq 0 \quad (i = 1, \dots, N), \quad p \gg 0; \\ & 0 \leq \frac{r}{1+r} \sum_i z_i^- \leq \bar{\mu}. \end{aligned}$$

The condition  $0 \leq \frac{r}{1+r} \sum_i z_i^- \leq \bar{\mu}$  arises from the following observation. The aggregate debt,  $(1+r)M$ , cannot exceed the total quantity of circulating money,  $\bar{\mu} + M$ , since default is not permitted in this paper. Hence, one must have:  $0 \leq r \leq \frac{\bar{\mu}}{M}$ . On the other hand, we have seen that, at least at equilibrium, this aggregate debt cannot exceed neither the global income earned from salings:  $(1+r)M \leq p \cdot \sum_i z_i^-$ . Condition  $0 \leq \frac{r}{1+r} \sum_i z_i^- \leq \bar{\mu}$  follows.

For the ease of notations, let us denote by  $\mathcal{A}$  the subset of  $(x, p, r) \in A \times \mathbb{R}_{++}^L \times \mathbb{R}_+$  such that  $0 \leq \frac{r}{1+r} \sum_i z_i^- \leq \bar{\mu}$ . We now claim that, if  $((x_i^*)_i, p^*, r^*)$  is a ME of  $\mathcal{L}$ , then it solves  $(\mathcal{P})$ . Clearly,  $a_i \cdot x_i^* = v_i(p^*, r^*)$  for every  $i$ . Hence, it suffices to show that

$$0 \geq \min_i [a_i \cdot x_i - v_i(p, r)] \quad \text{for every } (x, p, r) \in \mathcal{A}.$$

Suppose, on the contrary, that for some  $(x, p, r) \in \mathcal{A}$ , one has:  $a_i \cdot x_i > v_i(p, r)$  for every  $i$ . This implies  $p \cdot z_i^+ - \frac{1}{1+r} p \cdot z_i^- > \mu_i$ , every  $i$ . Summing over  $i$  gives (the first inequality arises from the feasibility of  $(z_i)_i$  and  $p \geq 0$ ):

$$\begin{aligned} p \cdot \sum_i z_i^- & \geq p \cdot \sum_i z_i^+ \\ & > \bar{\mu} + \frac{1}{1+r} p \cdot \sum_i z_i^-. \end{aligned}$$

The following contradiction follows:  $\frac{r}{1+r}p \cdot \sum_i z_i^- > \bar{\mu}$ .

*Step 3. Searching for convexity.*

Unfortunately, the objective function in  $(\mathcal{P})$  is not concave in general. As in Cornet (1989), we make a change of variable by defining, for  $q = (q_\ell)_\ell \in \mathbb{R}^L$ :

$$V_i(q, r) := v_i(e^{q_1}, \dots, e^{q_L}, r) \quad (i = 1, \dots, N).$$

Now, the following alternate programming problem is convex:

$$\begin{aligned} (\mathcal{Q}) : & \max_{(x_i), q, r} \min_{i=1, \dots, N} [a_i \cdot x_i - V_i(q, r)] \\ \text{subject to: } & \sum_i z_i \leq 0, \quad x_i \geq 0 \quad (i = 1, \dots, N), \quad q \in \mathbb{R}^L; \\ & \frac{r}{1+r} e^q \cdot \sum_i z_i^- \leq \bar{\mu}, \\ \text{with } e^q : &= (e^{q_1}, \dots, e^{q_L}), \quad q \in \mathbb{R}^L. \end{aligned}$$

Indeed, it follows from (24) and (26) that each  $V_i(\cdot, \cdot)$  can be expressed as a supremum of convex functions.

Let us prove that every solution of  $(\mathcal{Q})$  is a ME of  $\mathcal{L}$ . Let  $(x^*, q^*, r^*)$  be a maximum of  $(\mathcal{Q})$ . We first claim that:

$$u^* := \min_{i=1, \dots, N} [a_i \cdot x_i^* - V_i(q^*, r^*)] \geq 0.$$

Indeed, if  $(x^*, q^*, r^*)$  is a ME, then  $u^* = 0$ . As we know that ME exist,  $u^*$  yields as least 0.

It only remains to prove that every solution  $(x^*, q^*, r^*)$  of  $(\mathcal{Q})$  yields a ME. For this, all we have to check is that each  $x_i^*$  verifies  $i$ 's individual budget constraint. Since  $u^* \geq 0$ , one has, for every  $i$  (with  $p = (e^{q_1}, \dots, e^{q_L})$ ):

$$p^* \cdot z_i^{*+} - \frac{1}{1+r^*} p^* \cdot z_i^{*-} \geq \mu_i.$$

Summing over  $i$ , the same argument as at the end of Step 2 above enables to deduce from (27) that

$$p^* \cdot z_i^{*+} - \frac{1}{1+r^*} p^* \cdot z_i^{*-} = \mu_i \quad (i = 1, \dots, N).$$

As a conclusion, for  $(x^*, p^*, r^*)$ , to be a ME of  $\mathcal{L}$  is equivalent to  $(x^*, p^*, r^*)$  being a solution of  $(\mathcal{P})$  or  $(x^*, q^*, r^*)$  being a solution of  $(\mathcal{Q})$ .

Let denote  $\text{ME}_x$  the set of monetary equilibrium allocations:

$$\text{ME}_x := \{(x_i)_i \in (\mathbb{R}_+^L)^N \mid \exists(p, r) : (x, p, r) \in \text{ME}\},$$

and similarly, let  $\text{ME}_\pi$  denote the set of monetary equilibrium prices:

$$\text{ME}_\pi := \{p \in \mathbb{R}_+^L \mid \exists(x, r) : (x, p, r) \in \text{ME}\}.$$

A consequence of equivalence juste proven between ME and solutions of  $(\mathcal{Q})$  is that the two sets:

$$\text{ME} := \{(x, q, r) \mid (x, e^q, r) \in \text{ME}\} \text{ and } \text{ME}_\pi := \{q \mid e^q \in \text{ME}_\pi\}$$

are convex. Part (ii) then follows.

We know, from Dubey & Geanakoplos (2003a) (Theorem 4) that, in general concave economies,

$$\gamma(\omega) \geq r \geq \gamma(x^*),$$

for every ME allocation  $x^*$ . That is, gains-to-trade after trades occurred cannot exceed the interest rate, which itself cannot exceed initial gains-to-trade. Moreover, their Theorems 7 and 8 show that, when preferences are separable and strictly concave, then existence of a ME implies that  $\gamma(\omega) > r \geq \gamma(x^*)$  – which proves that GT is a necessary condition in their set-up. In the linear model of this Appendix, the situation is slightly different. For every  $x$  in the interior of the attainable set, it is clear that  $\gamma(x) = \gamma(\omega)$ . Hence, if  $(x, p, r)$  is a ME of  $(\mathcal{L}, \mu, M)$  with  $\bar{\mu}/M < \gamma(\omega)$ , since we know that  $r = \bar{\mu}/M$  at a ME, this means that  $x$  cannot be interior. In other words, in all the situations covered by part (i) of Proposition 4.1, the ME allocation will lie somewhere on the boundary of  $A$ . By contrast with the strictly concave setting, this, however, does not mean that no interior allocation can be a ME allocation or, equivalently, that GT is a necessary condition. Indeed, as shown by the next example (borrowed from Giraud & Tsomocos (2009)), a ME may exist (and be interior) even when  $\gamma(\omega) = r$ .

There are two agents and two commodities ( $N = L = 2$ ).  $\omega^1 = \omega^2 = (50, 50)$ ; private outside cash is  $\mu_1 = \mu_2 = \text{€}5$ ; inside money is  $M = \text{€}90$ . Utilities are  $u_1(x_1^1, x_2^1) = \frac{10}{75}x_1^1 + \frac{3}{25}x_2^1$ , and  $u_2(x_1^2, x_2^2) = \frac{3}{25}x_1^2 + \frac{10}{75}x_2^2$ . At the unique ME, prices are  $p_1 = p_2 = 1$ ; interest rate is  $r = \frac{1}{9}$ ; final allocations are  $x^1 = (50, -50)$ ,  $x^2 = (-50, 50)$ . Household 1 sells commodity 2 and buys 1. For this purpose, it spends its  $\text{€}5$  and buys 5 units of good 1. It also borrows  $\tilde{m}_1 = \text{€}45$  from the Bank, promising to repay  $(1 + r)\tilde{m}_1 = \text{€}50$ . This loan is spent to buy 45 additional units of good 1. Finally, agent 1 sells 50 units of good 2 to agent 2, and is able to repay the Bank. Traders' final utilities verify:

$$\frac{a_{11}}{p_1} = (1+r) \frac{a_{12}}{p_2} \quad \text{and} \quad \frac{a_{22}}{p_2} = (1+r) \frac{a_{21}}{p_1}.$$

Moreover,  $\gamma(\omega) = 1/9 = \gamma(x^*) = r = \bar{\mu}/M$ .

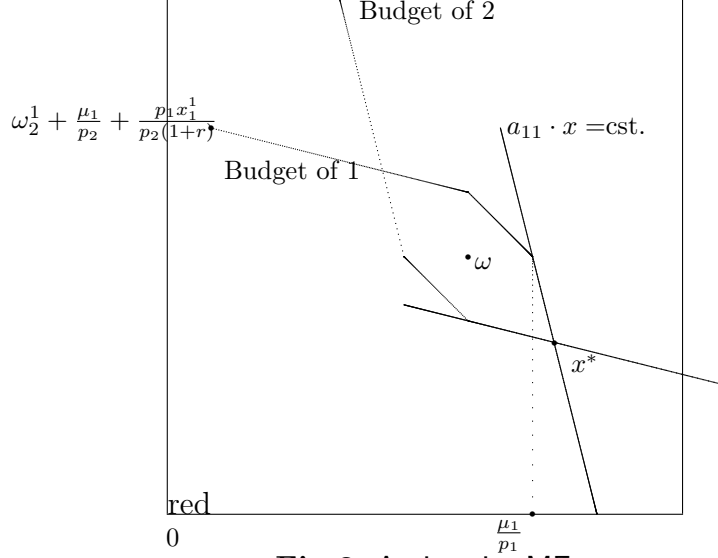


Fig 2. An interior ME

### 4.3 Uniqueness

In this section, we prove parts (iii) and (iv) of Proposition 4.1. Let us begin with the following:

**Lemma 4.1** *Let  $(p, q) \in \mathbb{R}_{++}^L$  and define  $s_\ell := \sqrt{p_\ell q_\ell}$  for every  $\ell$ . Then, for every  $r > 0$ ,*

$$\frac{1}{2}v_i(p, r) + \frac{1}{2}v_i(q, r) \geq v_i(s, r) \quad \text{for every } i,$$

*and the inequality is strict as soon as  $v_i(p, r) \neq v_i(q, r)$ .*

**Proof of the Lemma.** The inequality follows from the convexity of  $V_i(\cdot, r)$ . Suppose that, at  $(s, r)$ , household  $i$  is in case A. Choose commodity  $\ell$  so that

$$\hat{v}_i(s, r) = (\mu_i + \frac{1}{1+r} s \cdot \hat{\omega}_i(s, r)) \frac{a_{i\ell}}{s_\ell}. \quad (28)$$

Clearly,  $a_{i\ell} > 0$ . By the standard inequality between arithmetic and geometric means, one deduces that, for every good  $h$ :

$$\frac{1}{2} \left( \frac{\mu_i a_{i\ell}}{p_\ell L} + \frac{a_{i\ell}}{1+r} \frac{p_h \hat{\omega}_{ih}(s, r)}{p_\ell} + \frac{a_{i\ell} (\omega_i - \hat{\omega}_i(s, r))}{L} \right) + \frac{1}{2} \left( \frac{\mu_i a_{i\ell}}{p_\ell L} + \frac{a_{i\ell}}{1+r} \frac{q_h \hat{\omega}_{ih}(s, r)}{q_\ell} + \frac{a_{i\ell} (\omega_i - \hat{\omega}_i(s, r))}{L} \right)$$

$$\begin{aligned}
&\geq \sqrt{\frac{\mu_i^2 a_{i\ell}^2}{p_\ell q_\ell L^2}} + \sqrt{\frac{a_{i\ell}^2}{(1+r)^2} \frac{p_h q_h \hat{\omega}_{ih}^2(s, r)}{p_\ell q_\ell}} + \frac{a_i \cdot (\omega_i - \hat{\omega}_i(s, r))}{L} \\
&= \frac{\mu_i a_{i\ell}}{s_\ell L} + \frac{s_h \hat{\omega}_{ih}(s, r) a_{i\ell}}{(1+r) s_\ell} + \frac{a_i \cdot (\omega_i - \hat{\omega}_i(s, r))}{L}.
\end{aligned} \tag{29}$$

The end of this first part of the proof of our Lemma distinguishes between two cases: a) one of the inequalities above is strict for some commodity  $h$ ; b) all these inequalities are in fact equalities for every  $h$ . In the first case, summing up over  $h = 1, \dots, L$ , and recalling (24) and (28) gives:

$$\begin{aligned}
\frac{1}{2}v_i(p, r) + \frac{1}{2}v_i(q, r) &\geq \frac{1}{2} \left[ \left( \mu_i + \frac{p \cdot \hat{\omega}_i(s, r)}{1+r} \right) \frac{a_{i\ell}}{p_\ell} + a_i \cdot (\omega_i - \hat{\omega}_i(s, r)) \right] \\
&\quad + \frac{1}{2} \left[ \left( \mu_i + \frac{q \cdot \hat{\omega}_{ih}(s, r)}{1+r} \right) \frac{a_{i\ell}}{q_\ell} + a_i \cdot (\omega_i - \hat{\omega}_i(s, r)) \right] \\
&> \mu_i + \frac{s \cdot \hat{\omega}_i(s, r)}{1+r} \frac{a_{i\ell}}{s_\ell} + a_i \cdot (\omega_i - \hat{\omega}_i(s, r)) \\
&= v_i(s, r).
\end{aligned}$$

In the second case, b), one must have:

$$\begin{aligned}
v_i(p, r) &\geq \left( \mu_i + \frac{p \cdot \hat{\omega}_i(s, r)}{1+r} \right) \frac{a_{i\ell}}{p_\ell} + a_i \cdot (\omega_i - \hat{\omega}_i(s, r)) \\
&= \mu_i + \frac{s \cdot \hat{\omega}_i(s, r)}{1+r} \frac{a_{i\ell}}{s_\ell} + a_i \cdot (\omega_i - \hat{\omega}_i(s, r)) \\
&= v_i(s, r).
\end{aligned}$$

Similarly,  $v_i(q, r) \geq v_i(s, r)$ . But  $v_i(p, r) \neq v_i(q, r)$  then implies that  $\frac{1}{2}v_i(p, r) + \frac{1}{2}v_i(q, r) > v_i(s, r)$ .

Now, if household  $i$  is in case B at  $(s, r)$ , an almost identical calculus leads to the same conclusion. Details are left to the reader.

□

*Proof of Part (ii).*

The proof is essentially the same as for Theorem 2.2, except that the uniform approximation of  $\tau$ -localized utilities is no more needed, since utilities are already linear.

*Proof of Part (iii).* Obvious.

□